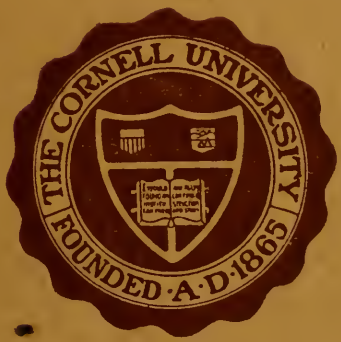
452

Hydrodyn. (Bathe)



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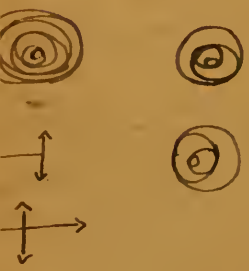
$M = \sum r_i \times p_i$

$\dot{M}_A = \sum r_i \times \dot{p}_i$

$= \sum r_i \times \sum f_{ij}$



Physics 452; Hydro- & Aero-dynamics. Prof. Bethe.

Books: Lamb: Classical; i.e. up to 1910. No modern development. Mathematical approach.

Collection on "Aerodynamic theory" - Prandtl, Taylor on
(7 vol) Viscous & Compressible fluids, respectively - entirely from physical viewpoint.

Collection "Modern devel. of fluid dynamics" - (hydro-).
(2 vol)

"Fluid mech. for hydraulic engineers" - more elementary

Prandtl & Tietjens -

Report to Nat'l Res. Coml - Bateman etc... Survey

"Secret" books - blue → Handbuch der Physik Vol VII good theoretical article.
green " " Expt. Physik. vol 4.

Feb. 2

Classical hydrodynamics: problems treatable by potential theory - i.e. 1) irrotational motion - $\nabla \times \underline{v} = 0$ If motion is everywhere irrotational, including boundaries, can show there should be no resistance to motion thru fluid; even including viscosity terms. Hence no real fluid is irrotational.

2) Stokes law: resistance ~ velocity for small velocities - No longer holds for large velocities.

3) Acoustics unlike 1) & 2), assumption of compressibility is made - also, all velocities << sound velocity. neglect 2nd order terms

Modern: Boundary layer theory of Prandtl. Takes care of fluid moving fast around bodies - neglect

certain terms near body; others elsewhere — Problem of body of arbitrary shape moving is still unsolved; even sphere isn't completely solved. Prandtl + Karman chief investigators.

Acoustics for large amplitude: shock waves. + velocities ≈ sound

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Classical theory can give useful results; e.g. in case of sphere moving thru fluid, get "potential flow" in front, + behind at sides; but not in wake. -(get vortices + turbulent flow).

Equations of motion found by Euler; in present form, by Stokes. (Stokes included viscosity).

No viscosity: state described by pressure — unique at each pt. (same in any direction) - $p = \frac{\text{force}}{\text{area}}$ - a scalar f'n of coord. + time $\underline{r}$ + t.

density ρ, a function of $(\underline{r} + t)$ also. In all simple cases, simple relation betwn ρ + p; Particularly simple case: density indep. of pressure (incompressible) e.g. water up to 100 atm. pressure. For gas, ρ definitely not constant. In many applications, can assume adiabatic changes for gases; ($p v^\gamma = \text{const}$ for ideal gas) $p \propto \rho^\gamma$. $\gamma = \frac{c_p}{c_v} = 1$ to $\frac{5}{3}$.

Also need $\underline{v}$ material velocity, with components u, v, w. $\underline{v}(\underline{r}, t)$.

One relation holds without consideration of forces — i.e. the continuity equation. If ρ is const in volume V, then amount of material is ρV; this changes with time; the change = amount which flows out, i.e. $\frac{\partial}{\partial t}\int_V \rho \, d\tau = -\int_S \rho v_n \, d\sigma = \int_V \nabla \cdot \rho \underline{v} \, d\tau = (\nabla \cdot \rho \underline{v}) V$

let $V \to 0$: $\boxed{\dot{\rho} + \nabla(\rho \underline{v}) = 0}$

this expresses conservation of matter.

For incompressible liquid, ρ = const and $\boxed{\nabla \cdot \underline{v} = 0}$

In general: $\boxed{\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho + \rho \nabla \cdot \underline{v} = 0}$

Euler equation of motion:

small volume element $d\tau$

$\rho\, d\tau$ = amt of material. want its acceleration

Newton: $(\rho\, d\tau)\frac{d\underline{v}}{dt}$ = Force

Forces of two kinds: "body" force like gravitation - prop. to mass of body. ∴ convenient to speak of force per unit mass. $\underline{F}$. Also force due to pressure: simple expression for uniform pressure on parallelepiped:

$F_x = (p_x - p_{x+dx})\, dy\, dz$

$= -\frac{\partial p}{\partial x}\, dx\, dy\, dz$

$= -\frac{\partial p}{\partial x}\, d\tau$

$\underline{F} = -\nabla p\, d\tau$

Hence equation becomes:

$\rho\, d\tau \frac{d\underline{v}}{dt} = \rho\, d\tau\, \underline{F} - d\tau (\nabla p)$

or $\boxed{\frac{d\underline{v}}{dt} = \underline{F} - \frac{1}{\rho}\nabla p}$.

Euler eq'n

Feb. 4. $\frac{d\underline{v}}{dt}$ is for a given element of matter. not same as was taken in deriving continuity eq'n. Will denote always by $\frac{\partial}{\partial t}$, derivation at a given pt. in space. $\underline{r}$. Will denote by $\frac{d}{dt}$, derivation for a given material element which moves with velocity $\underline{v}$ thru space.

Now $\frac{d\varphi}{dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial \varphi}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial \varphi}{\partial z}\frac{dz}{dt}$.

$= \frac{\partial \varphi}{\partial t} + \underline{v} \cdot \text{grad}\, \varphi$

Hence Euler eq'ns become: $\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \text{grad}\, \underline{v} = \underline{F} - \frac{1}{\rho}\nabla p$

x-component: $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$

Hydrostatics

Fluid at rest, $v = 0$

then $F = \frac{1}{\rho} \nabla p$.

In general F is known, p calculated from above eq'n.

A) If also ρ is constant, and F is a conservative force,

i.e. $F = -\nabla V$, then $\nabla(V + p/\rho) = 0$

and $V + p/\rho = \text{const.}$

If $\underline{F} = -g \underline{k}$ (gravity), then $V = gz$

whence $p = -\rho g z + \text{const.}$

(hydrostatic paradox)

1 atm. = 76 cm Hg = 1033 cm water = $1.012 \cdot 10^6$ dynes/cm^2

= 1.012 bar.

B) Suppose now ρ varies: (e.g. atmosphere). In general, $\rho = \rho(p)$

$$-g = \frac{1}{\rho} \frac{dp}{dz}.$$

then $-gz = \int \frac{dp}{\rho} = w - \text{const.}$

$w = \text{const} - gz$.

(Atmosphere, (nearly): $\rho = \frac{\rho_0}{p_0} p$

$w = \frac{p_0}{\rho_0} \int \frac{dp}{p} = \frac{p_0}{\rho_0} \ln p/p_0$

Whence $\ln p/p_0 = \text{const.} - \frac{\rho_0}{p_0} gz$

if $p = p_0$ at $z = 0$ (e.g. atm. at sea level)

then $\ln p/p_0 = -\frac{\rho_0}{p_0} gz$

$\therefore p = p_0 e^{-\frac{\rho_0}{p_0} gz}$.

c) Liquid rotating as rigid body. (incompressible)

$\underline{F} = -g\underline{k} + \omega^2 \underline{r}_1$ ($\underline{r}_1$ = comp. of $\underline{r}$ $\perp$ to axis of rotation)

then $V = gz - \frac{1}{2}\omega^2 r_1^2$

if axis is vertical, $r_1^2 = x^2 + y^2$, $\underline{r}_1 \perp \underline{k}$:

$V = gz - \frac{1}{2}\omega^2(x^2+y^2)$.

then $p = \text{const} - \rho V$

surfaces of constant pressure are paraboloids of revolution

free surface is then also a paraboloid.

Viscosity.

$\underline{v} = 0$ at wall. if wall is at rest; otherwise $\underline{v} = \underline{v}_{\text{wall}}$

Hence in general, there is a difference in velocity between various points of fluid - i.e. a velocity gradient. Hence there is a force - tangential along a boundary $\perp$ to velocity gradient - i.e. a shear stress - expect shear τ_{xy} to be proportional to velocity gradient $\frac{\partial u}{\partial y}$ if $u \parallel x$

write: $\tau_{xy} = \eta \frac{\partial u}{\partial y}$.

dimensions: $\left[\frac{\text{dyne}}{\text{cm}^2}\right] = [\eta]\left[\frac{\text{cm/sec}}{\text{cm}}\right]$

$\therefore [\eta] = \left[\frac{\text{dyne-sec}}{\text{cm}^2}\right] = \left[\frac{\text{gm}}{\text{cm-sec}}\right]$

water: $\eta \sim .01$ in this unit

Feb. 6 1942

$\tau_{xy} = \eta \frac{\partial v_x}{\partial y}$.

Flow thru cylindrical tube: Poiseuille.

if falling under own weight, then $p_2 - p_1 = \rho g h$

Consider a cylinder of fluid, of radius y;

What are forces acting on it? : pressure, top + bottom, viscous forces on sides. Net pressure force: $(p_2 - p_1)\pi y^2$.

assume shear constant over height of cylinder; then total shear force is given by: $\tau \cdot 2\pi h y$. hence $(p_2-p_1)\pi y^2 + \tau\, 2\pi h y = 0$

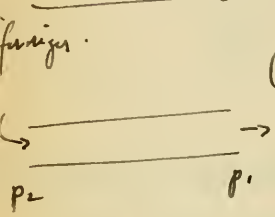
(τ is abs. magnitude) $\therefore \tau = -\frac{1}{2}(p_2-p_1)\frac{y}{h}$.

(Make tube horizontal) but $\tau = \eta \frac{\partial v}{\partial y}$.

foreign.

(v is velocity towards) $\therefore \frac{\partial v}{\partial y} = -\frac{1}{2}\frac{p_2-p_1}{\eta h}\cdot y$

τ pos. to right. $v = -\frac{1}{4}\frac{p_2-p_1}{\eta h} y^2 + \text{const.}$



but $v = 0$ at $y = r$,

$(p_2 > p_1)$ $\therefore v = \frac{p_2-p_1}{4\eta h}(r^2 - y^2)$

then total flow per unit time is:

$$\int_0^r v \cdot 2\pi y\, dy$$

$$= 2\pi \cdot \frac{p_2-p_1}{4\eta h} \int_0^r (r^2 - y^2)\, y\, dy \quad \Big\} = \frac{1}{2}\int_0^{r^2} u \frac{du}{2}$$

$$= \frac{\pi(p_2-p_1)}{2\eta h} \cdot \frac{1}{4} r^4$$

$$= \frac{\pi (p_2-p_1) r^4}{8 \eta h}.$$

From this can measure η with good accuracy ~ 1%.

liquids - (not so convenient for gases).

water: $\eta \sim 1.00$
air: $\eta \sim 1.0^{-4}$ c.g.s.

General velocity distribution: Viscous forces depend on nine quantities $\frac{\partial v_i}{\partial x_j}$. They form a dyadic – transform under rotations of axes like a dyadic. Can find particular combinations:

$(\nabla \cdot \underline{v})$ is invariant under rotations. (scalar).

$(\underline{\nabla} \times \underline{v})$ (curl $\underline{v}$) – transforms like an axial vector under rotation.

Can also form a tensor: (symmetrical)

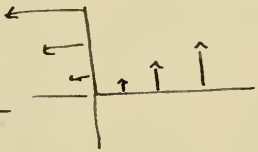
$$\begin{vmatrix} \frac{\partial v_x}{\partial x} - \frac{1}{3}\nabla\cdot v & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ & \frac{\partial v_y}{\partial y} - \frac{1}{3}\nabla\cdot v & \\ \text{etc.} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & \frac{\partial v_z}{\partial z} - \frac{1}{3}\nabla\cdot v \end{vmatrix}$$

(these five transforms like $P_{2m} e^{im\varphi}$ under rotation).

Physical significance: suppose $v_x = ax$: then fluid becomes extended in x-direction; likewise $\frac{\partial v_y}{\partial y}$ means extension in y-direction, $\frac{\partial v_z}{\partial z}$ in z-direction; to remain incompressible, $\nabla\cdot\underline{v}$ must vanish, i.e. extensions in one direction are compensated by ~~[illegible]~~ contractions in other directions. Hence sensible to separate this from other comp's.

Suppose $\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \neq 0$ other comp. of tensor $= 0$.

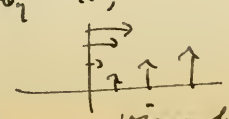
then $\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} = 0$; then $u_y = ax$, $v_x = -ay$.



corresponds to rotation of fluid as rigid body about origin. No viscous forces.

Next: $\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \neq 0$ then $v_y = ax$, $v_x = ay$

$\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0$



deforms ∟ to ∠

viscous forces [illegible]

Consider stress tensor.

Force across any boundary in a fluid is not in general normal to boundary; Stress is given by a symmetrical tensor whose space in general does not vanish. Diagonal components, σ_{xx} σ_{yy} σ_{zz},

Non diagonal: τ_{xy}, τ_{yz}, τ_{zx}.

If boundary has normal $\underline{n}$, then Force $\underline{p_n}$ acces plane | $p_{nx} = \sigma_{xx}\cos(n,x) + \tau_{xy}\cos(n,y) + \tau_{xz}\cos(n,z)$ etc.

Feb. 9, '42 | Assume medium isotropic: this will determine relation between stress tensor & deformation tensor almost completely.

Surface with normal $\underline{n}$; $\underline{p_n}$ = force across surface.

if $\underline{n}$ is in x-direction, call components of $\underline{p_n}$: p_{xx}, p_{xy}, p_{xz}.

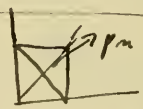
[or: σ_x τ_{xy} τ_{xz}. (elasticity notation)]

similarly $\underline{p_y}$ acting across boundary $\perp$ y direction.

comp. p_{yx} p_{yy} p_{yz}

or τ_{yx} σ_y τ_{yz} etc.

Consider equilibrium of:



Can deduce: $\underline{p_n} = \underline{p_x}\cos(n,x) + \underline{p_y}\cos(n,y) + \underline{p_z}\cos(n,z)$

i.e. $p_{nx} = p_{xx}$ " $+ p_{yx}$ " $+ p_{zx}$ "

$p_{ny} = p_{xy}$ " $+ p_{yy}$ " $+$ " "

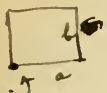
$p_{nz} =$ " " " "

[Can write as: $\underline{p_n} = \underline{\underline{p}} \cdot \underline{n}$.]

$\underline{\underline{p}}$ is stress tensor.

Can also deduce: $p_{xy} = p_{yx}$ etc.

Consider equilibrium against rotation:



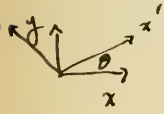
Moment $= \underline{r} \times \underline{F}$ $\qquad M_z = x F_y - y F_x$

then for equilibrium:

$$a \cdot p_{xy} \cdot bc - b \cdot p_{yx} \cdot ac = 0$$

$$\therefore p_{xy} = p_{yx}$$

i.e. stress tensor is symmetric.



rotate through angle θ.

then $p_{x'} = p_x \cos\theta + p_y \sin\theta$

$$p_{x'x} = p_{xx}\cos\theta + p_{yx}\sin\theta$$

component of $p_{y'}$ in direction x' is:

$$p_{x'x'} = p_{x'x}\cos\theta + p_{x'y}\sin\theta$$

$$= p_{xx}\cos^2\theta + 2p_{xy}\cos\theta\sin\theta + p_{yy}\sin^2\theta$$

likewise $p_{x'y'} = (-\tfrac{1}{2}p_{xx} + p_{yy})\cos\theta\sin\theta + p_{xy}(\cos^2\theta - \sin^2\theta)$

$$p_{y'y'} = p_{xx}\sin^2\theta - 2p_{xy}\cos\theta\sin\theta + p_{yy}\cos^2\theta.$$

$$\begin{cases} p_{xx} + p_{y'y'} = p_{xx} + p_{yy} \\ p_{x'x'} - p_{y'y'} = (p_{xx} - p_{yy})\cos 2\theta + 2p_{xy}\sin 2\theta \\ 2p_{x'y'} = (p_{xx} - p_{yy})\sin 2\theta + 2p_{xy}\cos 2\theta \end{cases}$$

sum of diag. comp. of stress tensor is unchanged under rotation; other comp. change to each other.

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3 dim: p_{xy}, p_{yz}, p_{zx} → transform into each other —

$(p_{xx} - p_{yy}), (2p_{zz} - p_{xx} - p_{yy})$ → " " " "

(like p_{xx}) $\frac{1}{3}(p_{xx} + p_{yy} + p_{zz})$ invariant under any rotation.

they transform like corresponding bilinear form : $x^2+y^2+z^2$ invr.;
other like xy, yz, zx; $x^2-y^2, 2z^2-x^2-y^2$,

Since stress tensor is symmetric, it can be related to symmetric part only of deformation tensor; (not to antisym i.e. curl $\underline{v}$ / part).
invariant part of stress must be related to invariant part of deformation
in fact, S-tensor of stress is proportional to S-tensor of deformation,
since rotation ——————— ? ?]

$$\therefore \quad p_{xy} = \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad \eta = \eta(\rho, T)$$

etc.
$$p_{xx} - p_{yy} = \eta\left(\frac{\partial v_x}{\partial x} - \frac{\partial v_y}{\partial y}\right)$$

etc.
$$p_{xx} - \tfrac{1}{3}(p_{xx}+p_{yy}+p_{zz}) = \eta\left(\frac{\partial v_x}{\partial x} - \tfrac{1}{3}\nabla\cdot\underline{v}\right).$$

$p = p(\rho) + \kappa\,\nabla\cdot\underline{v}$ ordinarily second term is small.
$\nabla\cdot\underline{v}$ very small for nearly incomp. fluid.

In general relation between κ & η.

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cb. 11'42 $\qquad p_{ik} + p\,\delta_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\nabla\cdot\underline{v}\,\delta_{ik}\right)$

$-p = \frac{1}{3}(\Sigma p_{ii}) \qquad p = p_0(\rho, T) + \kappa\,\nabla\cdot\underline{v}$

$$\therefore \quad p_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \delta_{ik}\left[p_0(\rho,T) + \left(\kappa + \tfrac{2}{3}\eta\right)\nabla\cdot\underline{v}\right]$$

$$\text{e.g.} \quad \sigma_x = 2\eta\frac{\partial v_x}{\partial x} - \left(p_0 + \left(\kappa + \tfrac{2}{3}\eta\right)\nabla\cdot\underline{v}\right)$$

$$\tau_{xy} = \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \quad \text{etc.}$$

Want to find eq'n of motion.

$p_n = \underline{p} \cdot \underline{n}$

Consider volume element: Total force due to stress is $\int \underline{p} \cdot \underline{n}\, d\sigma$

By Gauss' theorem $= \int d\tau\, \nabla \underline{p}$

(really apply to each comp. separately) $= (\Delta\tau)\, \nabla \cdot \underline{p}$

Force per unit volume $= \nabla \cdot \underline{p}$.

$\left[\text{e.g. } F_x = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right.$

Hence eq'n of motion becomes:

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{F} + \nabla \cdot \underline{\underline{p}}$$

x. comp:

$$\rho \frac{dv_x}{dt} = \rho F_x + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$= \rho F_x + \frac{\partial}{\partial x}\left[2\eta \frac{\partial v_x}{\partial x} - (p_0 + (\kappa + \tfrac{2}{3}\eta)\nabla \cdot \underline{v}) \right]$$

$$+ \frac{\partial}{\partial y}\left[\eta \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \right] + \frac{\partial}{\partial z}\left[\eta \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right) \right]$$

$$= \rho F_x + \eta \nabla^2 v_x + \eta \frac{\partial}{\partial x}(\nabla \cdot \underline{v}) - \frac{\partial p_0}{\partial x} + \frac{\partial}{\partial x}(\kappa + \tfrac{2}{3}\eta)\nabla \cdot \underline{v}$$

$$\rho \frac{dv_x}{dt} = \rho F_x + \eta \nabla^2 v_x - \frac{\partial p_0}{\partial x} - (\kappa + \tfrac{1}{3}\eta)\frac{\partial}{\partial x}\nabla \cdot \underline{v}$$

$$\rho \frac{d\underline{v}}{dt} = \rho \underline{F} + \eta \nabla^2 \underline{v} - \nabla p_0 - (\kappa - \tfrac{1}{3}\eta)\nabla(\nabla \cdot \underline{v})$$

$$= \rho \frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v}$$

or $$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v} = \underline{F} + \frac{\eta}{\rho}\nabla^2 \underline{v} - \frac{1}{\rho}\nabla p_0 - \frac{1}{\rho}(\kappa + \tfrac{1}{3}\eta)\nabla(\nabla \cdot \underline{v})$$

the last term is negligible usually; drop it. also drop subscript 0 from p_0. then have Navier–Stokes eq'ns. (∇v small).

the second order term $(\underline{v} \cdot \nabla)\underline{v}$ gives all the trouble.

if $\nabla^2 \underline{v}$ is neglected, the Euler equations are obtained.

$\nabla^2 \underline{v} = \nabla(\nabla \cdot \underline{v}) - \nabla \times (\nabla \times \underline{v})$

but since we have neglected term in $\nabla \cdot \underline{v}$, we have:

$$\dot{\underline{v}} + (\underline{v} \cdot \nabla)\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla \times (\nabla \times \underline{v})$$

I rotation of motion: $\nabla \times \underline{v} \equiv 0$; No viscous force.

Then $v = -\nabla \phi$; Classical hydrodynamics

this is why classical theory fails; because walls always require $v = 0$ somewhere; & require also a change in velocity (tangential) in direction normal to wall - i.e. $\text{curl}\, \underline{v} \neq 0$.

Similarity laws: if one solution is known, another can be found if scale is changed thereout.

$\frac{v^2}{l}$	$(\underline{v} \cdot \nabla)\underline{v}$ has dimension $\frac{cm^2}{cm\ sec^2} = \frac{cm}{sec^2}$
$\frac{\eta}{\rho}\frac{v^m}{l^2}$	$\frac{\eta}{\rho}\nabla^2 \underline{v}$ " " $\frac{gm}{cm\ sec} \cdot \frac{cm^3}{gm} \frac{cm}{sec\ cm^2} = \frac{cm}{sec^2}$

the quantity $R = \frac{\rho v l}{\eta}$ is dimensionless; ρ, η, v, are determined, need only pick a characteristic length l of the system. e.g. flow in tube, l could be radius,
v " - vel. at center.

R: Reynolds no. e.g. sphere moving in fluid: l could be diameter
v " - velocity of body.

e.g. fluid moving along plate: v could be velocity of fluid at large distance from plate., x distance travelled -

A solution of equations remains a solution if R is the same. (as far as $(\underline{v} \cdot \nabla)\underline{v} + \frac{\eta}{\rho}\nabla^2 \underline{v}$ are concerned) but p/ρ must be changed in same ratio as $(\underline{v} \cdot \nabla)\underline{v}$

Suppose all velocities multiplied by a :

" linear dimensions " " b .

Then if R is to be the same, $\frac{\eta}{\rho}$ must be changed in ratio ab .

$\frac{p}{\rho}$ changes in ratio a^2 . (really pressure differences)

Time changes in ratio b/a

Hence motion (e.g.) of high viscosity fluid with a real low-viscosity fluid with small dimensions ; or vice-versa low-velocity motion can be found with low-velocity fluid & diffnt viscosity etc .

Feb. 13 1911

Stokes eq'n . $\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\nabla)\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p - \frac{\eta}{\rho}\nabla\times\nabla\times\underline{v}$

Boundary cond: $v = 0$ at walls.

if R is kept united, get the same solution for a new problem with diffnt sizes, velocities, viscosities, etc .

R represents ratio of Eulers inertia term $\{\underline{v}\cdot\nabla\underline{v}\}$ to viscosity term $\left[\frac{\eta}{\rho}\nabla^2\underline{v}\right]$; hence if R is small, viscous term predominates ; $(\underline{v}\cdot\nabla)\underline{v}$ negligible ; if R is large, inertial term predominates, [illegible]

if R is small, get 2nd order, 1st degree equation, which is often soluble . (e.g. Stokes law) . valid up to $R \approx 10$.

Stresses $\sim \eta\,\nabla\cdot\underline{v} \sim \eta\frac{v}{l}$. $\therefore$ Force $\sim \eta\frac{vl^2}{l} = \eta vl$.

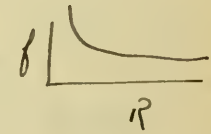
if R is large, viscous term still can't be neglected, esp. if we want to know about resistance to motion . Turns out that viscous term may be neglected over most of fluid, but not near body, where velocity changes quite rapidly .

$\frac{\partial v}{\partial t}$ gives pressures ~ acceleration of body;
if v is constant, only pressures are from Euler term $(v\cdot\nabla)v$, which gives pressure $\propto v^2$; hence reversing of all velocities gives same forces as before; hence must give no force (otherwise it could produce acceleration, which is absurd). ∴ Euler term gives no net resistance for uniformly moving body. For large R, classical flow (non viscous) except in a "boundary layer" surrounding solid bodies. For steady flow, neglecting body forces, get pressures $p \sim \rho v^2$, Force $\sim \rho v^2 l^2$; as long as body is not streamlined) [vortices are formed at back, + classical hydrodyn. doesn't hold.]

Hence $\frac{\text{Force (large } R)}{\text{Force (small } R)} \sim \frac{\rho v^2 l^2}{\eta v l} = \frac{\rho v l}{\eta} = R$. (actually about $R/10$)

For intermediate

In general: Resistance $= \rho v^2 l^2 f(R)$.
large R: $f(R) \sim 1$
small R: $f(R) = 1/R$.



Feb. 16

Water:
$\eta = .01032$ poise @ 20°C.
[poise] = $\left[\frac{\text{gm cm}}{\text{sec}}\right]$.

50°C : .00550
100°C : .00297

Hg : 20° : .0156
C_2H_5OH 20° : .0121
liquid CO_2 : .0007 at −30°.

organic fluids, many ~ .002 ~ .005 @ 20°
oils : ~ 1 ~ 10

Glacier : ~ 10^{10} (−10° : .87)

Gases : air @ 20° : $1.8 \cdot 10^{-4}$ (50° : 2.2)
η/ρ about 15 times larger for air than water

gases: η increases with T; liquids, decreases; mechanism

is different; gas viscosity: momentum transfer across boundary greater at higher temp., since momentum is greater; liquids, tendency of molecules to slip into holes left by others; slip easier at high T; $\therefore$ less viscosity. Liquid viscosity behaves roughly as $Ae^{+B/T}$. kB is activation energy.

Reynolds number: water, $\eta/\rho \sim .01$, $\rho/\eta \sim 100$.

$$R = 100 \cdot l \cdot v.$$

if $l \sim 1$ cm, $v \sim 1$ cm/sec, R is 100, which is large inertial term more important than viscosity term.

if $l \sim 1$ meter, $v \sim 10$ m/sec, $R \sim 10^7$ enormous.

To get small R, need, e.g. $l \sim 10^{-3}$ $v \sim 10^{-1}$ (dust particle falling thru liquid) $R \sim 10^{-2}$; Millikan oil drop: $R \sim 10^{-3}$ then can neglect inertial terms.

But for small particles, have to be careful about considering gas as a continuum; if particle size $\sim$ mean free path (10^{-5} cm) can't use hydrodynamical theory.

Return to hydrodyn. eq'ns.

Continuity: $\dot{\rho} + \nabla \cdot \rho \underline{v} = 0$

or: $\dot{\rho} + \underline{v} \cdot \nabla \rho + \rho \nabla \cdot \underline{v} = 0$

or: $\frac{d\rho}{dt} + \rho \nabla \cdot \underline{v} = 0$

$$\frac{1}{\rho}\frac{d\rho}{dt} = -\nabla \cdot \underline{v}$$

Eq'n of motion: $\dot{\underline{v}} + (\underline{v} \cdot \nabla)\underline{v} = \underline{F} - \frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \underline{v}$

Four eq'ns in 4 unknowns: $\underline{v}$, p.

ρ assumed uniquely related to p: "$p = p(\rho)$. equation of state".

[In general, $p = p(\rho, T)$, then need another eq'n. for T; comes from conductivity & energy conservation].

Present approximation: assume ρ constant.

then $\nabla \cdot \underline{v} = 0$.

2nd approx: $\rho(p)$ ρ a f'n of p alone; this is all that matters when motion becomes violent.

3rd approx: $\rho(p, T)$. (exact).

Incompressible fluid.

Exact Solutions: i.e. neglect no terms.

Assume fluid moving in x-direction; $v_y = v_z = 0$ $\cdot$ $v_x = u$.

Follows that u depends only on y & z : $u(y, z)$. [since $\frac{\partial u}{\partial x} = 0$ from continuity]

then $\frac{\partial u}{\partial t} + (\underline{v} \cdot \nabla) u = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\nabla^2 u$

$$\left.\begin{aligned} x: \quad & \frac{\partial u}{\partial t} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad \text{since } (\underline{v}\cdot\nabla)u = 0 \\ y: \quad & 0 = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ z: \quad & 0 = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}. \end{aligned}\right\}$$

If $\underline{F} = -\nabla V$, then $\frac{\partial V}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$

$\therefore V = -\frac{p}{\rho} + f(x)$

or $p = -\rho V + f(x)$.

$\therefore p = -\rho V + Ax$.

then $\frac{\partial u}{\partial t} = F_x - \frac{1}{\rho}\cdot(-\rho\frac{\partial V}{\partial x} + f'(x)) + \frac{\mu}{\rho}\nabla^2 u$

$= \frac{a}{\rho} f'(x) + \frac{\mu}{\rho}\nabla^2 u$

$\therefore$ ~~$\frac{a}{\rho}$~~ $f'(x) =$ not a f'n of $x = A$

$\therefore f(x) = Ax$

$\therefore \frac{\mu}{\rho} \nabla^2 u = \frac{\partial u}{\partial t} + \frac{A}{\rho}$

if flow is stationary, $\frac{\partial u}{\partial t} = 0$,

then $\nabla^2 u = A/\mu$. (a constant) .

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = A/\mu .$$

B.C. : $u = 0$ at boundary of cross section .

if we have a circle, get Poiseuille formula :

in cylindrical coordinates :

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = A/\mu = a .$$

$$\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = ar$$

$$r \frac{\partial u}{\partial r} = \frac{1}{2} a r^2 + b$$

$$\frac{\partial u}{\partial r} = \frac{1}{2} a r + b/r$$

$$u = \frac{1}{4} a r^2 + b \ln r + c$$

$$b = 0$$

$$\therefore u = \frac{1}{4} a (r^2 - R^2) .$$

require also u finite everywhere.

$b \neq 0$ only if no fluid at center -

$u = 0$ at $r = R$:

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If flow is non stationary : suppose $u = u(y)$ flow along plate in $x \cdot y$ plane ; let plate perform small oscillations then at boundary $u = A \sin \omega t$.

Equation is : $\frac{du}{dt} = \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}$ if no forces in x-direction

~~then $\omega = \frac{\mu}{\rho}$~~ if $u = u(y) e^{i\omega t}$

$$\therefore i\omega u(y) = \frac{\mu}{\rho} \frac{d^2 u(y)}{dy^2} .$$

$$\therefore u(y) = B e^{\sqrt{i\omega \rho/\mu}\, y}$$

Let $k = \sqrt{\frac{\omega\rho}{2\eta}}$ $\quad \sqrt{i}\,k = \pm(1+i)k$.

$\therefore u = Be^{\pm(1+i)ky}e^{i\omega t}$

$= Be^{-ky}e^{i(\omega t - ky)}$.

take sol'n with neg. real part (damping)

R.P. : $u = Be^{-ky}\cos(\omega t - ky)$.

(time lag ; motion of ~~boundary~~ at t, y, corresp. to motion of boundary at $t' = t - \frac{k}{\omega}y$.

$\frac{1}{k}$ gives thickness of layer affected by motion of plane; "boundary layer".

$\frac{1}{k} = \sqrt{\frac{2\eta}{\omega\rho}}$.

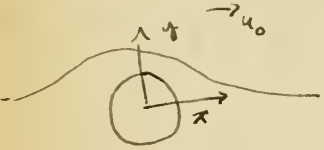
boundary layer thickness increases with η, decreases with ρ & ω. ; this is so in general.

Stokes' law

Small sphere thru fluid; R very small if defined by radius of sphere & its velocity. Then if radius a, $R = \frac{au_0\rho}{\eta} << 1$.

Neglect inertial term. No external forces. Stationary flow.

then: $\eta\nabla^2\begin{Bmatrix}u\\v\\w\end{Bmatrix} = \frac{dp}{dx}\begin{matrix}\}1\\\}4\\\}3\end{matrix}$.



sphere at rest

fl. at inf $= u_0$

$(v_0 = 0$

$w_0 = 0)$.

mean sphere get $v+w$ as well as u.

use both x, y, z & r, θ, φ (polar axis on x).

Expect: u indpnt of φ.

$u(x) = u(-x)$.

~~$v(y) = -v(-y)$~~ v { even in z / odd in x / odd in y.

w { odd in z / odd in x }

p vanishes at ∞.

p indep. of φ. p odd in x (gives resultant force on sphere)

p even in y + z.

$u = v\cdot a$

$\underline{v} = 0$ on sphere.

div $\underline{v} = 0$.

$$\eta \nabla^2 \underline{v} = \nabla p .$$

$$\therefore \eta \nabla^2 \nabla . v = \nabla . \nabla p = \nabla^2 p = 0 .$$

∴ pressure satisfies Laplace's eq'n.

$\therefore p = \frac{1}{r}$: not odd in x.

next : $p = \frac{\cos\theta}{r^2}$ OK. [or $p = \frac{P_n}{r^{n+1}}$. can show not possible]

[solutions $r^n P_n$ out because p vanishes at ∞]

$$\therefore p = \frac{\cos\theta}{r^2} = A \frac{x}{r^3} .$$

$$\frac{\partial p}{\partial x} = A\left[\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{x}{r}\right] = -A \frac{3x^2 - 1}{r^5} .$$

$$\frac{\partial p}{\partial y} = -A \frac{x}{r^4}\cdot\frac{y}{r} = -\frac{Axy}{r^5}$$

$$\frac{\partial p}{\partial z} = -A \frac{xz}{r^5} .$$

{Legendre f'ns of order 2 ÷ r^3}

$$\therefore \nabla^2 v_x = -\frac{2A}{\eta}\cdot\frac{1}{r^3}\cdot\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$$

$$= -\frac{2A}{\eta} \frac{P_2(\theta)}{r^3} .$$

assume $v_x = f(r) P_2(\theta)$.

in fact assume $v_x = \frac{\text{const}}{r} P_2(\theta)$.

$$\nabla^2 v_x(r,\theta) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r v_x) - \frac{1}{r^2}\Omega(v_x)$$

$$= -\frac{l(l+1)}{r^2} v_x \text{ const.} = -\frac{6\text{ const.}}{r^3} P_2$$

hence $v_x = \frac{A}{3\eta r}(3\cos^2\theta - 1)$ $\therefore$ const. $= \frac{A}{3\eta}$.

similarly $v_y = \frac{A}{3\eta r}\frac{3xy}{r^2}$ $\quad v_x = \frac{A}{3\eta r}\left(3\frac{x^2}{r^2}-1\right)$.

$v_z = \frac{A}{3\eta r}\frac{3xz}{r^2}$.

these satisfy conditions, except $\operatorname{div} \underline{v} = 0$.

∴ 1. add a term to v_x which makes $\operatorname{div}\underline{v}=0$

2. " terms to $\underline{v}$ they must also satisfy $\nabla^2 \underline{v} = 0$;

then if $\nabla^2 \underline{v}_2 = 0$, $\nabla \cdot \underline{v}_2 = 0$, then $\nabla \times \underline{v}_2 = 0$

$\therefore \underline{v}_2 = -\nabla\psi$.

also $\nabla^2\psi = 0$.

Feb. 20

Assumptions made so far, for Stokes law:

1. Stationary: $\frac{\partial \underline{v}}{\partial t} = 0$
2. Slow: neglect $\underline{v}\nabla\cdot\underline{v}$
3. No external forces.

With this, we get:

$\nabla^2 p = 0$

solution $p = A\frac{\cos\theta}{r^2}$ was considered appropriate.

Hence $\nabla^2 \underline{v} = \frac{1}{\eta}\nabla p$ can be found.

then found a particular integral: $\left\{\begin{array}{l} v_x = +\frac{A}{6\eta r}\left(3\frac{x^2}{r^2}-1\right) \\ v_y = + \;\ldots\left(\frac{3xy}{r^2}\right) \\ v_z = + \;\ldots\left(\frac{3xz}{r^2}\right)\end{array}\right\}$.

but these don't satisfy $\nabla\cdot\underline{v}=0$;

hence must add solution of $\nabla^2\underline{v}=0$, which will make $\nabla\cdot\underline{v}=0$.

Now $v_\rho = \frac{v_y y + v_z z}{\sqrt{y^2+z^2}} = \frac{A}{6\eta r}\cdot\frac{3x}{r^2}\sqrt{y^2+z^2}$.

$= \frac{A}{6\eta r}\cdot 3\cos\theta\sin\theta$.

also $\begin{cases} v_z = v_x\cos\theta + v_\rho\sin\theta \\ v_\theta = -v_x\sin\theta + v_\rho\cos\theta \end{cases}$

whence $v_r = \frac{A}{6\eta r}\cos\theta(3\cos^2\theta - 1 + 3\sin^2\theta)$

$= \frac{A}{6\eta r} \cdot 2\cos\theta$.

$v_\theta = \frac{A}{6\eta r}\sin\theta(-3\cos^2\theta + 1 + 3\cos^2\theta)$

$= \frac{A}{6\eta r}\sin\theta$.

$\therefore \nabla\cdot\underline{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta \sin\theta)$

$= \frac{A}{6\eta r^2}\cdot 2\cos\theta + \frac{A}{6\eta r^2}\cdot 2\cos\theta = \frac{A}{6\eta r^2}(4\cos\theta)$.

$\therefore$ need to add to $\underline{v}$ a velocity field with $\nabla\cdot\underline{v}' = -$ ↖ .

(Can do this with: $v_x' = \frac{\text{const}}{r}$; $v_y = v_z = 0$

then $\nabla\cdot\underline{v}' = -\frac{\text{const}}{r^2}\cos\theta$

$\therefore$ const $= \frac{4A}{6\eta}$

$\therefore v_x' = \frac{4A}{6\eta r}$

$\therefore$ must add to v_r: $v_x'\cos\theta$ $\quad\therefore$ New $v_r = \frac{A}{6\eta}\cdot 6\cos\theta$

$= \frac{A}{\eta}\cdot\cos\theta$.

add to v_θ: $v_x'\sin\theta$

$\therefore$ New $v_\theta = \frac{A}{6\eta}\sin\theta[1-4]$

$= -\frac{A}{2\eta}\sin\theta$.

Now these satisfy eq'n. of motion + continuity condition;

Need to satisfy boundary cond.; $\therefore$ solution which we add satisfy $\nabla^2\underline{v}_2 = 0$, $\nabla\cdot\underline{v}_2 = 0$

instant $\nabla\times\underline{v}_2 = 0$ $\quad\therefore \underline{v}_2 = -\nabla\phi$.

this ψ must give correct velocity asymptotically.

for large r: $\underline{v} = \begin{Bmatrix} u_0 \\ 0 \\ 0 \end{Bmatrix}$.

Can do this with $\psi = -u_0 r + \text{terms vanishing at } r = \infty$.

$= \cos\theta\,(-u_0 r) + \ldots\ldots$

In fact, can satisfy everything if we take also the other sol'n of Stokes's eq'n $\perp$ to $\cos\theta$: $\psi = \cos\theta\left[-u_0 r + B/r\right]$.

then $v_{2r} = -\frac{\partial \psi}{\partial r} = -\cos\theta\left[-u_0 - \frac{2B}{r^3}\right]$.

$v_{2\theta} = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = -\sin\theta\left[u_0 - \frac{B}{r^3}\right]$.

Adding these to original sol'n $\underline{v}_1$, we get:

$v_r = v_{1r} + v_{2r} = \cos\theta\left[+u_0 + \frac{2B}{r^3} + \frac{A}{\eta r}\right]$

$v_\theta = v_{1\theta} + v_{2\theta} = -\sin\theta\left[u_0 - \frac{B}{r^3} + \frac{A}{2\eta r}\right]$.

with A + B can satisfy the remaining conditions at sphere:

i.e. $v_r = 0 = v_\theta$ at $r = a$:

this easily done by $\frac{A}{\eta} = -\frac{3}{2} u_0 a$

$B = \frac{1}{4} u_0 a^3$.

$$\therefore \boxed{\begin{aligned} v_r &= u_0 \cos\theta \left(1 + \tfrac{1}{2}\tfrac{a^3}{r^3} - \tfrac{3}{2}\tfrac{a}{r}\right) \\ v_\theta &= -u_0 \sin\theta \left(1 - \tfrac{1}{4}\tfrac{a^3}{r^3} - \tfrac{3}{4}\tfrac{a}{r}\right) \end{aligned}}$$

If potential flow alone had been considered, would have had only one constant, hence can satisfy only one B.C.

e.g. $v_r = 0$ at sphere; get flow along sphere —

Potential flow would be completely determined by this: for $\nabla^2\psi = 0$, + if $\frac{\partial \psi}{\partial r}$ is given at sphere, + at ∞, ψ is uniquely determined: this gives $v = 0$ at front and back of sphere ("stagnation pts")

$\therefore\ v_\theta = -\frac{3}{2}u_0$ at $\theta = \pi/2$, $r = a$; speeds up, as it must.

~~[illegible]~~

at surface of sphere $\frac{\partial v_r}{\partial \theta}$, $\frac{\partial v_\theta}{\partial \theta}$ are both zero;

also $\frac{\partial v_r}{\partial r} = 0$ at surface.

{this is really required by continuity eq'n, $\nabla \cdot \underline{v} = 0$, since

$\nabla \cdot v = \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \text{something} \cdot r \cdot 0 = 0 \quad \therefore \frac{\partial v_r}{\partial r} = 0$.

$\therefore$ Normal velocity has vanishing normal derivative at solid. This is generally true. $\therefore$ Stresses can only contain $\frac{\partial v_r}{\partial \theta}$; this is the only one not $= 0$.

Suppose terms in P_n had been taken for solution of $\nabla^2\psi = 0$.

$r^n P_n$ are out, (for $n \geq 2$) since that gives $v \sim r$ or worse at ∞. $\therefore$ only use $r^{-n-1} P_n$ $\psi \sim \frac{C}{r^{n-1}} P_n$; then can't make $\frac{\partial \psi}{\partial r} = 0$ at $r = a$ unless $C = 0$. Similarly, if higher terms are added in pressure, get another constant D; then one B.C. could be satisfied, e.g. $v_r = 0$ at $r = a$; but not any more; hence this is out. For $n = 0$, same thing; $\psi \sim A + \frac{B}{r}$, $v \sim \frac{B}{r^2}$, + can't satisfy conditions at $r = a$.

Hence only terms $n = 1$ occur, as we have used.

Feb. 23 Force exerted on sphere by fluid.

$\therefore$ need τ_{rx} : force in x-direction across boundary $\perp r$ (per cm²)

we had $\tau_{rx} = \frac{x}{r}\sigma_x + \frac{y}{r}\tau_{xy} + \frac{z}{r}\tau_{rz}$.

we had : $\sigma_x = 2\eta\frac{\partial v_x}{\partial x} - p$

$\tau_{xy} = \eta\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$

$\tau_{xz} = \eta\left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right)$

$\therefore \tau_{rx} = \eta\frac{\underline{r}}{r}\cdot\left(\nabla v_x + \frac{\partial \underline{v}}{\partial x}\right) - \frac{x}{r}p$.

$= \eta\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) - \frac{x}{r}p$.

$v_x = v_r\cos\theta - v_\theta\sin\theta$

$\therefore \tau_{rx} = \eta\left(\cos\theta\frac{\partial v_r}{\partial r} - \sin\theta\frac{\partial v_\theta}{\partial r} + \frac{\partial v_r}{\partial x}\right) - \frac{x}{r}p$

at $r=0$, $\frac{\partial v_r}{\partial r} = 0 = \frac{\partial v_r}{\partial \theta}$;

$\therefore \tau_{rx} = -\eta\sin\theta\frac{\partial v_\theta}{\partial r}\Big|_{r=a} - \frac{x}{r}p\Big|_{r=a}$

$= -\eta\cdot\sin\theta\cdot -u_0\cdot\sin\theta\cdot\frac{3}{2}\cdot\frac{1}{a} - \frac{x}{r}p\Big|$

$= \frac{3}{2}\frac{\eta u_0}{a}\sin^2\theta - p\cos\theta$

$= \frac{3}{2}\frac{\eta u_0}{a}\sin^2\theta + \frac{3}{2}\frac{\eta u_0}{a}\cos^2\theta$

$= \frac{3}{2}\frac{\eta u_0}{a}$!

$v_\theta = -u_0\sin\theta\left(1 - \frac{1}{4}\frac{a^3}{r^3} - \frac{3}{4}\frac{a}{r}\right)$

$\therefore \frac{\partial v_\theta}{\partial r} = -u_0\sin\theta\left(+\frac{3}{4}\frac{a^3}{r^4} + \frac{3}{4}\frac{a}{r^2}\right)$

$p = A\frac{\cos\theta}{r^2}$ $\quad A = -\frac{3\eta}{2}u_0 a$

$p\big|_{r=a} = -\frac{3}{2}\eta u_0 a\cdot\frac{\cos\theta}{a^2}$

Hence pressure + viscous forces give force independent of θ !

$\therefore$ Total force $= \frac{3}{2}\frac{\eta u_0}{a}\cdot 4\pi a^2$ $\boxed{= 6\pi\eta u_0 a}$!

Stokes law

If only this force acts on sphere, it comes to rest; if it travels, say, several diameters, flow is quasi-stationary, & can be treated using above

If an external force acts, reach a final velocity,

given by $F = 6\pi\eta u_0 a$.

Suppose particle of density ρ_m are suspended in fluid of density ρ_f.

then $F = \frac{4\pi}{3} a^3 g(\rho_m - \rho_f) = 6\pi\eta u_0 a$

and terminal velocity $u_0 = \frac{4\pi a^3 g(\rho_m - \rho_f)}{3 \cdot 6\pi\eta a} = \frac{2}{9} a^2 (\rho_m - \rho_f) \frac{g}{\eta}$

if $g \sim 1000$
$\eta \sim 10^{-2}$ (H_2O)
ρ

$u_0 = \frac{2}{9} \cdot \frac{10^3}{10^{-2}} (\rho_m - \rho_f) a^2$

$= 2 \cdot 10^4 (\rho_m - \rho_f) a^2$

if $a \sim 10^{-3}$ cm., $u_0 \sim .02$ cm/sec if $\rho_m - \rho_f \sim 1$.

How far can this be applied? $R = \frac{u_0 \cdot 2a \rho_f}{\eta}$

$= \frac{4}{9} a^3 \frac{\rho_f (\rho_m - \rho_f) \cdot g}{\eta^2} \ll 1$

Air: $\rho_f \sim 10^{-3}$
$\eta = 2 \cdot 10^{-4}$
$\rho_m \sim 1$

"little raindrops"

$\therefore \frac{4}{9} \cdot 25 \cdot 10^6 \times 10^7$
$\therefore a^3 < 10^{-7}$

damn small raindrops!

If $\rho_f = 1$, $\rho_m - \rho_f = 1$,

$R = \frac{4}{9} \cdot a^3 \cdot \frac{10^3}{10^{-4}} < 1$

$a^3 < 2 \cdot 10^{-7}$
$a < \sqrt[3]{2} \cdot 10^{-2}$
$a \lesssim 5 \cdot 10^{-3}$
barely visible
$u_0 \sim \frac{1}{2}$ cm/sec

Proof that solution we had is unique.

$\eta \nabla^2 \underline{v} = \nabla p$ $\nabla \cdot \underline{v} = 0$

$\therefore \underline{v} \cdot \eta \nabla^2 \underline{v} = \underline{v} \cdot \nabla p$

$\therefore \eta \int (v_x \nabla^2 v_x + v_y \nabla^2 v_y + v_z \nabla^2 v_z) d\tau = \int \underline{v} \cdot \nabla p \, d\tau$

$\therefore \eta \left[\int_S \left(v_x \frac{\partial v_x}{\partial n} + v_y \frac{\partial v_y}{\partial n} + v_z \frac{\partial v_z}{\partial n} \right) d\sigma - \int_V \left((\nabla v_x)^2 + (\nabla v_y)^2 + (\nabla v_z)^2 \right) d\tau \right] = \int \underline{v} \cdot \nabla p \, d\tau$

Consider:

infinite sphere.

at surface of body, $v_x = v_y = v_z = 0$

$\therefore$ surface integral vanishes at body

Suppose there are 2 sol'ns: $\underline{v_1} p_1$ $\underline{v_2} p_2$.

then $\underline{v} = \underline{v_1} - \underline{v_2}$ $p = p_1 - p_2$ is also a sol'n.

also $\underline{v} = 0$ at ∞ since v_1 & v_2 correspond to same rel. stress.

then for $\underline{v}$, surface integral vanishes also at infinite sphere

(we had that a sol'n must be of form $p \sim r^n P_n$

also $\psi \sim r^n P_n$

Hence velocity comp's: $v_r = v_{r_0} + \frac{v_{r_1}}{r} f(\theta) + \cdots$

but we excluded higher powers; $\therefore v_r \sim 1/r$

$\frac{\partial v}{\partial r} \sim 1/r^2$

$\therefore v \frac{dv}{dr} \sim 1/r^3$

$\therefore$ surface integral vanishes)

Likewise $\int \underline{v} \cdot \nabla p \, d\tau$

$= \int \nabla \cdot p \underline{v} \, d\tau - \int p \nabla \cdot v \, d\tau$

but $\nabla \cdot v = 0$; $\therefore = \int_S p v_n \, d\sigma = 0$ on surface of body, also on infinite sphere;

Hence $\int \left((\nabla v_x)^2 + (\nabla v_y)^2 + (\nabla v_z)^2 \right) d\tau = 0$

$\therefore \nabla v_{x_{y,z}} = 0$ everywhere.

$\therefore v_x, v_y, v_z = 0$ everywhere since they vanish at ∞.

$\therefore \underline{v_1} = \underline{v_2}$

Hence solution is unique.

Feb. 25) Improvements due to Oseen. Large distances.

Can't use perturbation theory, since inertial terms which were neglected are larger than the viscous terms!

We had $u \sim u_0\left(1+\frac{a}{r}f(\theta)\right)$.

then $\nabla^2 u \sim \frac{u_0 a}{r^3} f(\theta)$ $\quad \nabla u \sim \frac{u_0 a}{r^2} g(\theta)$.

inertial terms: $\rho(v\cdot\nabla)v \sim \rho\frac{u_0^2 a}{r^2}$

viscous ~~terms~~: $\sim \eta\frac{u_0 a}{r^3}$ $\quad$ large r, viscous < inertial.

ratio $\frac{\text{inertial}}{\text{viscous}}$: $\frac{\rho u_0^2 a}{\eta r^2 u_0 a}\cdot r^3 \sim \rho\frac{u_0 r}{\eta}$.

$\therefore$ for large enough R, our solution can't be correct

for large r, velocity is nearly u_0.

$\therefore$ approximate $\underline{v}$ by asymptotic value in inertial terms.

$(\underline{v}\cdot\nabla)v = u_0\frac{\partial}{\partial x}\underline{v}$.

then get: $\rho u_0\frac{\partial \underline{v}}{\partial x} = -\nabla p + \eta\nabla^2\underline{v}$

Let $\underline{v} = u_0\underline{i} + \underline{v}'$.

divide $\underline{v}'$ into 2 parts: one eliminates pressure:

$\underline{v}' = \underline{v}_1 + \underline{v}_2$.

Assume: $\left\{\begin{array}{l}\rho u_0\frac{\partial \underline{v}_1}{\partial x} = -\nabla p \\ \underline{v}_1 = -\nabla\varphi_1 \\ \nabla\cdot\underline{v}_1 = 0\end{array}\right\}$ (from these, $\nabla^2\underline{v}_1 = 0$).

then $\frac{\rho u_0}{\eta}\frac{\partial \underline{v}_2}{\partial x} = \nabla^2\underline{v}_2$

Now $\nabla^2\varphi_1 = 0$

also $\rho u_0\frac{\partial}{\partial x}(-\nabla\varphi_1) = -\nabla p$

$\therefore \rho u_0\frac{\partial\varphi_1}{\partial x} = p$ $\quad$ (choose zero of p as that const. vanishes)

also $\nabla^2 p = 0$

$\underline{v_2}$: Let $\rho \frac{u_0}{\eta} = 2k$

if $v_2 = +\nabla \chi$, and $\nabla^2 \chi = 2k \frac{\partial \chi}{\partial x}$

then $\nabla^2 \underline{v_2} = 2k \frac{\partial \underline{v_2}}{\partial x}$ is satisfied;

but not $\nabla \cdot \underline{v_2} = 0$.

Now $\nabla \cdot \underline{v_2} = \nabla^2 \chi = 2k \frac{\partial \chi}{\partial x}$

$\therefore$ need to subtract something with divergence $-2k \frac{\partial \chi}{\partial x}$.

$\therefore$ Let $\underline{v_2} = \nabla \chi - 2k \chi \underline{i}$

then $\nabla \cdot \underline{v_2} = \nabla^2 \chi - 2k \frac{\partial \chi}{\partial x} = 0$.

Can solve by $\chi = e^{kx} \psi$

then $\frac{\partial \chi}{\partial x} = e^{kx}\left(\frac{\partial \psi}{\partial x} + k\psi\right)$.

$$\frac{\partial^2 \chi}{\partial x^2} = e^{kx}\left(k^2 \psi + 2k \frac{\partial \psi}{\partial x} + \frac{\partial^2 \psi}{\partial x^2}\right)$$

$$\nabla^2 \chi = k^2 \psi + 2k e^{kx} \frac{\partial \psi}{\partial x} + e^{kx} \nabla^2 \psi.$$

$$\therefore \nabla^2 \chi - 2k \frac{\partial \chi}{\partial x} = \nabla^2 \psi - k^2 \psi = 0$$

$$(\nabla^2 - k^2)\psi = 0$$

simplest sol'n: $\psi = \frac{1}{r} e^{\pm kr}$ throw out $+$.

(other sol'n's with $P_{nm}^{(\theta,\varphi)}$) $\therefore$ take $\psi = \frac{1}{r} e^{-kr}$

$$\therefore \chi = \frac{A}{r} e^{-k(r-x)}.$$

this sol'n has $\nabla \cdot v = 0$ except at origin;

calculate fluid crossing sphere of radius r:

$$r^2 \int v_r \cdot \sin\theta \, d\theta \, d\varphi \, dr$$

$$v_r = \frac{\partial \chi}{\partial r} - 2k\chi \cos\theta$$

make r very small: then $e^{ikr} \sim 1$, and 2nd term is small comp. with first; $\therefore \int = -4\pi A$; i.e. a negative source (sink) at centre of sphere. Hence need to add a solution with equal sink; hence $\underline{v}_1$ must to this.
$\therefore$ choose a ϕ_1' compensating this such: $\phi_1' = A/r$.

Can add also $\phi_1'' = \frac{Bx}{r^3} = \frac{B\cos\theta}{r^2}$, to satisfy B.C. (has right symmetry) Hence complete soln:

$$\underline{v} = u_0 \underline{i} - 2k\chi \underline{i} + \nabla\chi - \nabla\phi_1$$
$$= (u_0 - 2k\chi)\underline{i} + \nabla\left(\frac{A}{r} e^{-ik(r-x)} - \frac{A}{r} - \frac{Bx}{r^3}\right)$$

On surface of sphere: (small kr) [$R = \frac{\rho u_0 a}{\eta}$ still small i.e. $2ka$ small]

For small kr: $e^{-ik(r-x)} = 1 - ik(r-x)$

then $\underline{v} = (u_0 - 2k\chi)\underline{i} + \nabla\left(\frac{-Aik(r-x)}{r} - \frac{Bx}{r^3}\right)$

$\chi \sim \frac{A}{r}(1 - ik(r-x))$ $\nabla\left(Aik(1-\cos\theta) - \frac{B\cos\theta}{r^2}\right)$.

Now $v_r = (u_0 - 2ik\chi)\cos\theta + \frac{2B\cos\theta}{r^3}$.

$= \left(u_0 - \frac{2ikA}{r}\right)\cos\theta + \frac{2B\cos\theta}{r^3} = \cos\theta\left[u_0 - \frac{2ikA}{r} + \frac{2B}{r^3}\right]$

and $v_\theta = \left(u_0 - \frac{2ikA}{r}\right)(-\sin\theta) - \frac{\sin\theta}{r}\left(Aik - \frac{B}{r^3}\right)$.

~~$= -\sin\theta\left[u_0 - \frac{2ikA}{r}\right]$~~

$= -\sin\theta\left[u_0 + \frac{ikA}{r} - \frac{B}{r^3}\right]$

at $r = a$, $v_r = v_\theta = 0$: $\therefore Aik = 3/4\, u_0 a$

$B = 1/4\, u_0 a^3$.

$\therefore$ small r soln same as before:

$v_r = u_0 \cos\theta \left[u_0 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\frac{a^3}{r^3} \right]$
$v_\theta = -u_0 \sin\theta \left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\frac{a^3}{r^3} \right]$. same as before - but holds only near sphere!

this gives some resistance, in this approximation.
Can convert by a power series in ka $\quad (4ka = R)$.

then get Resistance $= 8\pi\mu a u_0 \left(1 + \frac{3}{16}R - \frac{19}{1280}R^2 + \frac{71}{20,480}R^3 \cdots\right)$

Feb. 27

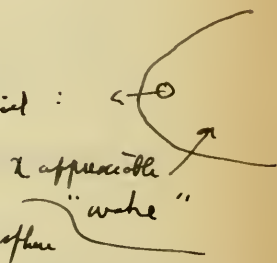
We had: $\underline{v} = u_0 \underline{i} + \nabla(\chi - \varphi) - 2k\chi \underline{i}$

found $\chi = \frac{A}{r} e^{-k(r-x)}$ $\qquad k = \frac{u_0 \rho}{2\eta} = \frac{R}{4a}$

$\varphi = \frac{A}{r} + \frac{B\cos\theta}{r^2}$

$Ak = \frac{3}{4} u_0 a$, $B = \frac{1}{4} u_0 a^3$.

$\chi \to 0$ for large r except in x-direction.
χ is appreciable for, say, $k(r-x) = 1$
this is a paraboloid:



outside wake, $\underline{v} - u_0 \underline{i} = -\nabla\varphi$

large r: $-\nabla\varphi = \frac{A}{r^2}$; source at sphere
which means that fluid ahead of body diverges - makes room for body. $\leftarrow \infty$

acts like source of strength $A = \frac{3}{4}\frac{u_0 a}{k} = \frac{3}{4}\frac{u_0 a}{u_0 \rho} 2\eta = \frac{3}{2}\frac{a\eta}{\rho}$

i.e. independent of velocity of body

in wake: large distances: $r \sim x$, $\nabla(\chi - \varphi) \sim \frac{1}{r^2}$
$=$ term $2k\chi$ becomes important: $\sim \frac{1}{r}$.

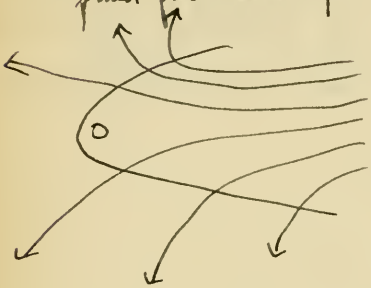
$-2k\chi$ is most important term:

gives velocity in ~~x~~ x direction $= -2k\frac{A}{r}e^{-\kappa(r-x)}$.

$= -2\cdot\frac{3}{4}\frac{u_0 a}{r}e^{-\kappa(r-x)}$

$= -\frac{3}{2}u_0\frac{a}{r}e^{-\kappa(r-x)}$.

fluid follows body.



pressure $= \rho u_0\frac{\partial\varphi}{\partial x}$

$= \rho u_0\left(\left(-\frac{x}{r^3}\right)A - \frac{B}{r^3}(3\cos^2\theta-1)\right)$

1st term small behind body; pushes fluid out in front, sucks in fluid from behind; 2nd term is sym. about plane $x=0$. no resultant force

---

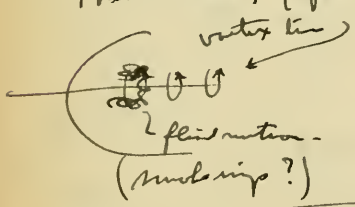
$(\nabla\times\underline{v})_y = -2k\frac{\partial\chi}{\partial z} = 2k\frac{z}{r}e^{-\kappa(r-x)}(1+\kappa r)$ gives vortices

$(\nabla\times\underline{v})_z =$ .. $\frac{y}{r}$

$(\nabla\times\underline{v})_x = 0$. $\nabla\times\underline{v}$ is $\perp$ to x; in fact circular about x

can draw "vortex lines"; circles with x-axis as center.

then ~~velocity of fluid~~ fluid rotates about these lines.



Case of cylinder has also been solved — Here we add a sol'n $\log r/a$ to the symmetric solution; but this diverges. Using Oseen's method, can get a solution like sphere — χ is a Bessel f'n of imaginary argument., get a constant asymptotic sol'n

Resistance $= \dfrac{4\pi\eta u_0 l}{\log\frac{\eta}{\rho u_0 a}+\frac{1}{2}+\log 4-C}$

March 6
1942
A.D.
?-(3)

It would be nice to use potential flow, since it is simplest; however, need to consider deviations. The greatest deviations, we found for sphere, are right around body, & in wake. So potential flow can be used in front & sides (roughly) of body, but not ~~around~~ on boundary & in wake.

Let $\underline{\omega} = \nabla \times \underline{v}$ = "vorticity".

In two dimensions : $v_x \; v_y \; (x, y, t)$.

Continuity : $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}$$

Eq'n of motion: $\rho \frac{\partial v_x}{\partial t} + \frac{\partial v_x}{\partial x} v_x + \frac{\partial v_x}{\partial y} v_y = -\frac{\partial p}{\partial x} + \eta \nabla^2 v_x$

(similar one for v_y).

Multiply (1st by $\partial/\partial y$, second by $\partial/\partial x$, & subtract:

Get : $\rho \frac{\partial \omega}{\partial t} + v_x \frac{\partial \omega}{\partial x} + v_y \frac{\partial \omega}{\partial y} = \eta \nabla^2 \omega$ (using $\nabla \cdot \underline{v} = 0$)

Eq'n for heat conduction in medium moving with velocity $\underline{v}$.

if $\eta \to$ heat conductivity, $\rho \to$ heat capacity.

$$\rho \frac{d\omega}{dt} = \eta \nabla^2 \omega.$$

Like body being hot & heating up fluid; only fluid which comes near body gets hot, i.e. part which goes into wake; high temp. means vorticity –

Suppose ω is infinite at $x = y = 0$ (in such a way that its integral over $x = y = 0$ is finite). $\omega = 0$, $x \neq 0$, $y \neq 0$. How will this "vortex filament" spread?

[bring a certain amt of heat to origin at $t = 0$, let it spread –

then $\rho \frac{\partial \omega}{\partial t} = \eta \nabla^2 \omega$;

sol'n is : $\omega = \frac{A}{t} e^{-\frac{r^2}{4\mu t}}$ $\qquad \mu = \eta/\rho$.

i.e. heat (vorticity) spreads according to this law —

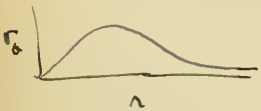
$$\int_0^a \omega \cdot 2\pi r\, dr = \int (\nabla \cdot \underline{v})_n\, d\sigma = \int_0^{2\pi} v_\theta\, r\, d\theta = 2\pi r v_\theta = \Gamma \text{ (circulation)}$$

$$\therefore \quad v_\theta = \frac{1}{2\pi r} \cdot 2\pi \int_0^a \frac{A}{t} e^{-\frac{r^2}{4\mu t}} r\, dr$$

$$= \frac{A}{rt} \int_0^a e^{-\frac{r^2}{4\mu t}} r\, dr = \frac{2\mu A}{r} \left(1 - e^{-\frac{r^2}{4\mu t}}\right)$$

if $r \gg \sqrt{4\mu t}$, $\quad v_\theta \sim \frac{1}{r}$. $\quad$ i.e. Γ is constant.

if $r \ll \sqrt{4\mu t}$, $\quad v_\theta \sim r$ $\quad$ i.e. $\Gamma \sim r^2$.



as t increases, vortex filament increases ;
max velocity position moves out, & value of " " decreases.

(Now we see it was OK. to neglect quadratic terms, since $(\underline{v} \cdot \nabla)\underline{v} = 0$ since grad $\underline{v}$ is radial & v is tangential.

$R = \frac{\rho u_0 l}{\eta} \gg 1$.

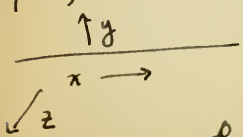
Hip velocity.

Fluid flowing past long flat plate

Potential flow : plate makes no difference, doesn't satisfy $v_x = 0$ at

Expect potential flow OK. far from plate ;

Near plate, a "boundary layer" of increasing thickness with different sol'n.



Expect thickness of layer << length of plate if suff. great R.

Let δ = thickness of boundary layer ($\delta(x)$).
l = length of plate.

Continuity: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$.

$$v_y = \text{const} - \int_0^y \frac{\partial v_x}{\partial x}\, dy$$

but $v_y = 0$ at $y = 0$, $\therefore$ const $= 0$.

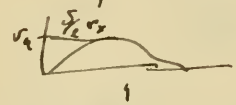
$$v_y = -\int_0^y \left(\frac{\partial v_x}{\partial x}\right) dy$$

$$\approx \frac{v_x}{l} \cdot \delta$$

Hence in B.L.,

$$v_y \approx v_x \cdot \frac{\delta}{l}.$$

i.e.



Eq'n of motion: $\nabla^2 v_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}$;

$$\approx \frac{v_x}{l^2} + \frac{v_x}{\delta^2} \approx \frac{v_x}{\delta^2}.$$

i.e. viscosity term comes from change of $\perp$ comp. of velocity

hence neglect $\frac{\partial^2 v_x}{\partial x^2}$.

$$\rho\left[v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right]$$

$$\approx \rho\left[v_x \frac{v_x}{l} + v_x \cdot \frac{\delta}{l} \frac{v_x}{\delta}\right] \approx \rho \frac{v_x^2}{l}.$$

Assume flow don't change rapidly: $\rho \frac{\partial v_x}{\partial t} \approx \rho \frac{v_x^2}{l}$.

τ = time in which flow changes by its own magnitude. i.e. $\frac{v_x}{\tau} \approx \frac{v_x^2}{l}$

i.e. $\tau \approx \frac{l}{v_x}$

| left | right |

i.e. time for fluid to flow length of plate.

then $\eta \frac{v_x}{\delta^2} \approx \rho \frac{v_x^2}{l}$

i.e. $\delta^2 \approx \frac{\eta l}{\rho v_x}$

i.e. $\frac{\delta}{l} \approx \sqrt{\frac{\eta}{\rho l v_x}} = \frac{1}{\sqrt{R}}$

ratio width to length of B.L. $\approx \frac{1}{\sqrt{R}}$.

From the heat (diffusion) eq'n, we know also that distance

to which heat (vorticity) goes is $r = \sqrt{4\mu t}$; $= \sqrt{\frac{4\mu l}{v_x}}$

$= \sqrt{\frac{4\eta l}{\rho v_x}}$

same as before

March 9 Eq'n of motion for y-component:

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right)$$

negligible again.

$$p(x,\delta) - p(x,0) \sim \frac{\partial p}{\partial y}\cdot\delta \sim \rho v_x \frac{v_x}{l}\cdot\delta \sim \rho v_x^2 \left(\frac{\delta}{l}\right)^2$$

$\therefore$ change in p $\perp$ to surface thru B.L. may be neglected.

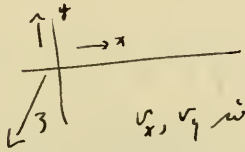
Continuity eq'n: $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$

Let $v_x = \frac{\partial \psi}{\partial y}$, $v_y = -\frac{\partial \psi}{\partial x}$; ψ: stream function.

(not a potential!).

(don't need $\nabla \times \underline{v} = 0$)

flat plate again:



$y = 0$.

v_x, v_y will depend on $\frac{y}{\delta}$ ($\delta(x)$ = width of B.L. at x)

Let $\delta(x) = \sqrt{\frac{\eta x}{\rho u_0}} = \sqrt{\mu x/u_0}$

If zero viscosity, p = const., v = const.; $\therefore$ zero th approx: $\frac{\partial p}{\partial x} = 0$.

Assume stationary flow, $\frac{\partial v_x}{\partial t} = 0$:

then $\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = \eta \frac{\partial^2 v_x}{\partial y^2}$.

Introduce $\psi = \sqrt{\mu u_0 x}\, f(\xi)$ $\xi = y/\delta$.

$\mu = \eta/\rho$ then $v_x = \sqrt{\mu u_0 x}\, f'(\xi)\cdot\frac{1}{\delta} = u_0 f'(\xi)$

$v_y = -\sqrt{\mu u_0 x}\, f'(\xi)\cdot -\frac{y}{\delta^2}\cdot\frac{1}{2}\frac{1}{\sqrt{x}}\cdot\sqrt{\mu/u_0} - \frac{1}{2}\sqrt{\mu u_0}\frac{1}{x^{1/2}}\cdot f(\xi)$

$\ldots f' \frac{y u_0}{\ldots} - \frac{1}{2}\mu f \ldots = -\frac{\sqrt{\mu u_0 x}}{\ldots}(\xi f' - f)$

Need $f'(\xi) \to 1$ for $\xi \to \infty$.

then diff. eqn becomes

$$\frac{\partial v_x}{\partial x} = u_0 f''(\xi) \cdot -\frac{\xi}{2x}$$

$$\frac{\partial v_x}{\partial y} = u_0 f''(\xi) \cdot \frac{\xi}{y} = u_0 f''(\xi) \cdot \sqrt{\frac{u_0}{\mu x}}$$

$$\frac{\partial^2 v_x}{\partial y^2} = u_0 \left[f'''(\xi) \cdot \frac{u_0}{\mu x} \right]$$

$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu/\rho \frac{\partial^2 v_x}{\partial y^2}$ becomes:

$$u_0 f'(\xi) \cdot u_0 f''(\xi) \cdot -\frac{\xi}{2x} + \sqrt{\frac{u_0}{\mu x}} \left(\xi f' - f \right) \cdot u_0 f'' \sqrt{\frac{u_0}{\mu x}} = \mu \cdot u_0 f''' \frac{u_0}{\mu x}$$

or: $u_0^2 \left[f' f'' \left(-\frac{\xi}{2x}\right) + \frac{1}{2x} \left(\xi f' - f \right) f'' \right] = u_0^2 \cdot \frac{1}{x} \cdot f'''$

$\therefore \; -\xi f' f'' + \xi f' f'' - f f'' = 2 f'''$

~~$-2\xi f' f'' - f f'' = 2 f'''$~~

$\therefore$ assumption correct (about v_x)

$$\therefore \; f f'' + 2 f''' = 0$$

B.C.: $f' = 1$ as $\xi \to \infty$

for $\xi = 0$: $v = 0$, i.e. $f' = 0$

$f = 0$.

Can eqn in series: $f = \sum \frac{A_n}{n!} \xi^n$

$$f'' = \sum \frac{A_n}{(n-2)!} \xi^{n-2} \qquad f''' = \sum \frac{A_n}{(n-3)!} \xi^{n-3}$$

$$\therefore \left(\sum \frac{A_n}{n!} \xi^n \right) \left(\sum \frac{A_n}{(n-2)!} \xi^{n-2} \right) + 2 \sum \frac{A_n}{(n-3)!} \xi^{n-3} = 0$$

Terms at $A_{1,2,3,4} = 0$; $A_5 = -\frac{A_2^2}{2}$ $A_6 = 0$ $A_7 = 0$,

also at $A_{8 \text{ on}}$

get : $f = \sum_0 c_n \frac{\alpha^{n+1}}{(3n+2)!} \left(-\frac{1}{2}\xi\right)^{3n+2}$

$c_0 = 1$
$c_1 = 1$
$c_2 = 11$
$c_3 = 375$ etc.

Converges. $\alpha = A_2$?

$c_5 \sim 4{,}000{,}000$!

For large ξ: $f_1 = \gamma\xi - \beta$ would satisfy eq'n;

if $f = f_1 + f_2$, $f_2 << f_1$ for large ξ

then $f_1 f_2'' + 2f_2''' = 0$

$$\frac{f_2'''}{f_2''} = -\frac{1}{2} f_1 = -\frac{1}{2}(\gamma\xi - \beta)$$

$$\therefore \log f_2'' = ~~-\frac{1}{2}\left(\gamma\frac{\xi^2}{2} - \beta\xi\right)~~$$

$$= -\frac{1}{4}(\gamma\xi - \beta)^2$$

Can fit solution

correctly by numerical methods —

$$f_2'' = \varepsilon\, e^{-\frac{1}{4}(\gamma\xi - \beta)^2}.$$

want $\gamma = 1$

March 11 $\quad f f'' + 2f''' = 0$

can multiply f by a, ξ by $1/a$

Hence if $f(\xi)$ is a sol'n, then $g(\xi) = a f(a\xi)$ is also.

then $g'(\xi) = a$ [illegible]

Numerical sol'n done by Tupper;

$\alpha = .332$ to fit correct

sol'n at ∞.

$$g g'' = a f(a\xi) + a^3 f''(a\xi)$$

$$2g''' = 2a^4 f(a\xi).$$

and $g'(\xi) = a^2 f'(a\xi)$

$\rightarrow a^2$ as $\xi \rightarrow \infty$

$$f = \xi - \beta + \varepsilon\, e^{-\frac{1}{4}(\xi-\beta)^2}$$

$\beta = 1.73$

$\varepsilon = .231$

ξ	$f'(\xi)$
0	0
.4	.133
1	.330
2	.630
3	.846
4	.955
5	.9966

for $\xi \sim 5$, reach asymptotic value of velocity.

From this can calculate force on plate —

$$\text{Shear} = \tau = \eta\left(\frac{\partial v_x}{\partial y}\right). \qquad \left(\frac{\partial v_y}{\partial x} = 0\right)$$

$$\left.\frac{\partial v_x}{\partial y}\right|_{y=0} = u_0 f''(\xi)\cdot\frac{\partial \xi}{\partial \eta} = u_0 f''\cdot\frac{1}{\sqrt{\frac{\eta x}{\rho u_0}}} = u_0^{3/2}\cdot\sqrt{\frac{\rho}{\eta x}}\cdot\alpha.$$

$$\therefore \tau = \alpha\cdot u_0^{3/2}\cdot\sqrt{\frac{\rho\eta}{x}}.$$

$$\text{Total force (per unit length)} = \int_0^l \tau\,dx = \alpha u_0^{3/2}\sqrt{\rho\eta}\cdot\frac{x^{1/2}}{1/2}\Big|_0^l$$

$$= 2\alpha u_0^{3/2}\sqrt{\rho\eta l}.$$

Diffn: Resistance coefficient C_D:

$$\text{Force} = \tfrac{1}{2} C_D \rho u_0^2 l$$

$$\text{Here } C_D = \frac{4\alpha u_0^{3/2}\sqrt{\rho\eta l}}{\rho u_0^2 l} = 4\alpha\sqrt{\frac{\nu}{\rho u_0 l}} = \frac{4\alpha}{\sqrt{R}} \ll 1.$$

(This all for large R)

Consider fluid crossing section from $y=0$ to $y=y_1$: $\int_0^{y_1} u\,dy$.

Compare with pot. flow: $\int_0^{y_2} u_0\,dy$

where y_2 is such that get same flow for both

then point at y_2, before plate, goes to y_1, at plate.

Beyond $\xi \approx 5$, $y_1 - y_2$ is constant.

$$\int_0^{y_1} u\,dy = \int_0^{y_2} u_0\,dy \;\&;\; \int_0^{y_1}[u_0-(u_0-u)]\,dy = \int_0^{y_2} u_0\,dy = u_0 y_2$$

$$\therefore u_0 y_1 - \int_0^{y_1}(u_0-u)\,dy = u_0 y_2$$

$$y_1 - y_2 = \frac{1}{u_0}\int_0^{y_1}(u_0-u)\,dy.$$

but if y_1 is large enough,

$u_0 - u \approx 0$, hence can replace by ∞:

$$y_1 - y_2 = \frac{1}{u_0}\int_0^{\infty}(u_0-u)\,dy$$

$$= \int_0^{\infty}[1-f'(\xi)]\,d\xi\cdot\delta$$

since $f' \to \beta$, $\quad y_1 - y_2 = \delta \lim_{\zeta \to \infty} (\zeta_1 - f(\zeta_1)) = \beta\delta$.

$$\delta^* = y_1 - y_2 = \beta\delta \qquad \beta \sim 1.23$$

B.L. potential flow beyond $\sim 3\delta^*$.

Flows about arbitrary shapes:

pot. flow: $v = 0$ at ends, v max at top & bottom;
$\therefore$ pressure max at front & back, min. at top & bottom

$p = p_0 - \frac{1}{2}\rho v^2$ behind, reverse flow develops in B.L.;

Hence get:

i.e. vortices form.

B.L. "separation" from surface

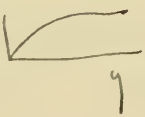
fluid takes place along any shape body.

To streamline: make pressure increase slowly

At separation point $\frac{\partial v_x}{\partial y} = 0$ Hence $v_x \frac{\partial v_x}{\partial x}$, $v_y \frac{\partial v_x}{\partial y}$ are zero,

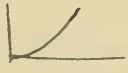
get only: $-\frac{\partial p}{\partial x} + \eta \frac{\partial^2 v_x}{\partial y^2} = 0$ from eq of η motion.

(In fact this holds on surface of body) $\quad \frac{\partial^2 v_x}{\partial y^2} = \frac{1}{\eta}\frac{\partial p}{\partial x}$. before S.P.

if p is decreasing, get 

if p is increasing, get

or:

at separation pt.: 

if $\frac{dp}{dx}$ is [illegible]

after S.P.

March 13 | Bernoulli theorem:

Potential flow: $\underline{v} = \nabla\varphi$ $\quad \nabla\times\underline{v} = 0$ $\quad \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}$ etc.

& stationary flow: $\rho(\underline{v}\cdot\nabla)\underline{v} = -\nabla p + \eta\nabla^2\underline{v}$ but no viscosity (potential flow)

$\therefore (\underline{v}\cdot\nabla)\underline{v} = -\frac{1}{\rho}\nabla p$

$\therefore \rho v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z} = -\frac{\partial p}{\partial x}$

or $\rho v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial x} + v_z\frac{\partial v_z}{\partial x} = -\frac{\partial p}{\partial x}$

or $\rho\,\underline{v}\cdot\frac{\partial\underline{v}}{\partial x} = -\frac{\partial p}{\partial x}$

$\therefore \rho\frac{1}{2}\frac{\partial v^2}{\partial x} = -\frac{\partial p}{\partial x}$

$\therefore \frac{1}{2}\rho v^2 = -p + p_0$

or $p = p_0 - \frac{1}{2}\rho v^2$.

at a boundary, if u is [ruptured] velocity, $p = p_0 - \frac{1}{2}\rho u^2$.

Physically, Bernoulli's th. says that work done by pressure goes into kinetic energy.

If $\mathrm{curl}\,\underline{v} \neq 0$, this is true if you follow a certain part of fluid along a streamline. Pot. flow: p_0 same for all stream lines;

Non-pot. flow: p_0 different for each stream line

Turbulence: volume elements are broken down, then can't use it at all.

[" starts $\sim R \sim 10^5$]

Hence can get pressure distrib. from velocity distrib. (pot flow)

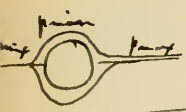
$\nabla\cdot\underline{v} = 0$; $\therefore \nabla^2\varphi = 0$; $v_n = \frac{\partial\varphi}{\partial n} = 0$ at boundary.

$\nabla^2\varphi = 0$, with B.C. $\frac{\partial\varphi}{\partial n}$ at walls & φ known at ∞ is often soluble; e.g. sphere (which we did). We get $v_\theta = u_0\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right)\sin\theta$

$\varphi = u_0\cos\theta\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)$ $\quad v_r = u_0\cos\theta\left(1 - \frac{a^3}{r^3}\right)$

In this case, stagnation pt. at $\theta = 0$, $r = a$; $p = p_0$;

at $\theta = \pi/2$, $v_\theta = \frac{3}{2} u_0$, $\therefore p_{min} = p_0 - \frac{1}{2}\rho\sigma_\theta^2$



Only 1st part of this distrib. has sense.

At separation point, $y = 0$, $\frac{\partial u}{\partial y} = 0$ if u is velocity in tangential direction (direction of x - coord. on body) origin at stag. pt. y is coord. $\perp$ to body.

Pot. flow, $u_1 = a_1 x + \cdots$ (expand) Suppose body symmetrical about direction of incident fluid; then velocity is odd f'n;

Hence $u_1 = a_1 x + a_3 x^3 + \cdots$; $p = p_0 - \frac{1}{2}\rho a_1^2 x^2 - \cdots x^4 - \cdots x^6 \cdots$

and $\frac{\partial p}{\partial x} = -\rho a_1^2 x - \cdots x^3 \cdots$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = a_1^2 x + 4a_1 a_3 x^3 + (6a_1 a_5 + 3a_3^2)x^5 \cdots$$

Introduce $\psi(x, y)$;

$u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$; take $\psi = \psi_1(y)\cdot x + \psi_3(y)\, x^3 + \psi_5(y)\, x^5 + \cdots$

$\psi_h = 0$ at $y = 0$ $\quad \frac{\partial \psi_h}{\partial y} = 0$ at $y = 0$

$\frac{\partial \psi_h}{\partial y} = a_h$ at $y = \infty$

Can put this in eq'n of motion, with same assumptions as before, & get finally:

$p_1 = a_1^2$

$p_3 = 4a_1 a_3$

etc.

[Eq'n of motion practically the same if thickness of B.L. is << radius of curvature]

x: $\psi_1'^2 - \psi_1\psi_1'' = p_1 + \mu\psi_1'''$

x^3: $4\psi_1'\psi_3' - 3\psi_1''\psi_3 - \psi_1\psi_3'' = p_3 + \mu\psi_3'''$

etc. Write $\psi_1(y) = \sqrt{a_1\mu}\, f_1\left(y\sqrt{\frac{a_1}{\mu}}\right)$.

then get: $f_1'^2 - f_1 f_1'' = 1 + f_1'''$.

argument $y\sqrt{\frac{a_1}{\mu}} \sim \frac{y}{\sqrt{\eta}\, dx} \sim$ thickness of B.L.

this equation is integrated numerically subject to B.C.:

$f_1 = 0$, $f_1' = 0$ at $\eta = 0$; $f_1' = 1$ at $\eta = \infty$ $\cdot [\psi' = a_1 \text{ at } \eta = \infty]$.

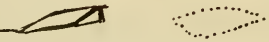
Solution looks like one we had before;

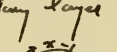
Then get ψ_3; then $\psi = \psi_1 x + \psi_3 x^3$

and $\left(\frac{\partial u}{\partial \eta}\right)_0 = \left(\frac{\partial^2 \psi}{\partial \eta^2}\right)_0 = a_1^{3/2} x f_1'' + a_3 a_1 x^3 f_3''$

equate to 0 to find separation pt. (a_3 is negative).

(Hiemenz)



April 6 We had discussed separation of boundary layer by expansion in terms of x: distance from stagnation pt  ; $u + v$, (velocity along τ $\perp$ to surface) expressible in terms of stream function ψ: $u = \frac{\partial \psi}{\partial \eta}$, $v = \frac{\partial \psi}{\partial x}$

$\psi = x \psi_1 + x^3 \psi_3 + \cdots$; $\psi_1 \left(\frac{y}{\sqrt{\nu/a_1}}\right)$, a_1 from: $\psi_1(x) = a_1 \xi x + a_3 x^3 + \cdots$

velocity profile changes with x:

when $\frac{\partial u}{\partial \eta} = 0$ at $\eta = 0$, separation pt

Behind separation pt., can't do anything by this method - have vortices & diff

Case also investigated change with time - Blasius Zs. für Ph. & M., VI 1908

Boltze - 3 dim. case -

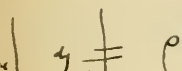
Approximate methods, easier to handle are desirable.

Kármán

Consider flat plate again.

friction force τ measures momentum transfer from fluid to plate. $\tau = \frac{dM}{dx}$ x = coord along plate

M = total loss of mom. since beginning.

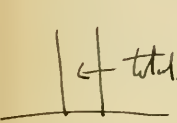


$\rho u\, dy$ assessment mom.

$\therefore$ mom = $\rho u dy \cdot u$

loss of momentum = $\rho u dy (u_1 - u)$.

Total $= \int_0^{u=\delta} \rho u\, dy (u_1 - u)\, dy$ $\qquad \tau = \eta \left(\frac{\partial u}{\partial y}\right)_0$



← total mom going out = mom. coming in

(Kármán : assume a reasonable distrib ; then find thickness of B.L.

assume $u \neq u_1$ only if $y < \delta(x)$ assume : in B.L., $u = u(y/\delta) = u(\xi)$

Then $u = u_1 f(\xi)$ when $f(\xi) \to 1$ for $\xi \to \infty$ $\therefore M = \rho u_1^2 \delta \int_0^1 f(1-f)\, d\xi$

$= \rho u_1^2 \delta \alpha$

$\tau = \frac{\eta u_1}{\delta} \left(\frac{\partial f}{\partial \xi}\right)_0 = \frac{\eta u_1}{\delta} \beta$

$\therefore \tau = \rho u_1^2 \alpha \frac{d\delta}{dx} = \beta \eta \frac{u_1}{\delta}$.

Thickness of boundary layer $\sim x^{1/2}$

$\therefore \delta \frac{d\delta}{dx} = \frac{\beta}{\alpha} \cdot \frac{\eta}{\rho u_1}$

$\delta^2 = \frac{2\beta}{\alpha} \frac{\eta x}{\rho u_1}$

$\therefore \delta = \sqrt{\frac{2\beta}{\alpha}} \sqrt{\frac{\eta x}{\rho u_1}}$.

Need only $\beta \neq \alpha$.

guess a suitable f :

B.C. on f : $f = 0$ at $\xi = 0$

$f = 1$ at $\xi = 1$

try $f = \xi$ then $\alpha = \int_0^1 \xi(1-\xi)\, d\xi = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ and $\sqrt{\frac{2\beta}{\alpha}} = \sqrt{\frac{2 \cdot 1}{1/6}}$

$\beta = \int_0^1 1 \cdot d\xi = 1$ $\therefore \delta = 2\sqrt{3} \sqrt{\frac{\eta x}{\rho u_1}}$

get $\tau = \beta_0 \cdot \eta u_1 = \frac{2}{2\sqrt{3}\sqrt{\frac{\eta x}{\rho u_1}}} = \frac{1}{2\sqrt{3}} \cdot \rho^{\frac{1}{2}} \eta^{\frac{1}{2}} u_1^{\frac{3}{2}} x^{-\frac{1}{2}}$

$2M = \rho u_1^2 \delta_2 = 2\rho u_1^2 \sqrt{\frac{\eta x}{\rho u_1}} \cdot 2\sqrt{3} \cdot \frac{1}{6} = \left(\frac{2}{\sqrt{3}}\right) \sqrt{\eta \rho u_1^3 x}$.

1.155

Displacement of flow in y direction $\delta^* = \frac{\int (u_1 - u)\, dy}{u_1}$ correct : 1.328.

$= \frac{\delta u_1 \int_0^1 (1-f)\, d\zeta}{u_1}$

$= \frac{1}{2} 2\sqrt{3} \sqrt{\frac{\eta x}{\rho u_1}} \cdot \int_0^1 (1-f)\, d\zeta$

$= \sqrt{3} \cdot \sqrt{\frac{\mu x}{u_1}}$

exact theory gave $1.72 \sqrt{\frac{\mu x}{u_1}}$.

$\therefore$ ok.

Forces hydrodyn eq'ns:

$\eta \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} + \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} \right)$

$\to 0$

at surface of plate, $\zeta = 0$;

$\therefore f'' = 0$ at $\zeta = 0$;

$\therefore$ take $f = \frac{3}{2}\zeta - \frac{1}{2}\zeta^3$

also want $f' = 0$ at $\zeta = 1$, for smooth join –

this gives $2M = 1.29$ – within 3%.

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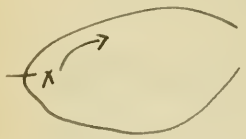
sk8 Try same method on curved surface

$u_1(x)$ = velocity along x. (potential flow)

$\tau = \frac{dM}{dx}$



Consider section of fluid between plane x & $x + dx$

If flow is non-stationary, there is a change of mom : $\frac{d}{dt}\left(\int_0^h \rho u\, dy\right) dx = \rho dx \int_0^h \frac{du}{dt}\, dy$

Net outflow of mom. thru sides $= \rho dx \frac{d}{dx}\left(\int_0^h u^2\, dy\right)$

[amount] fluid leaving $= \rho u dy$, mom. $= \rho u^2 dy$].

(These are both supplied by plate). [h is in pot. flow region]

Amount leaving thin upper surface : $(\rho v\, dx)\cdot u_1$

Now $\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$ $\therefore$ $v = -\int_0^h \frac{\partial u}{\partial x}\, dy$

Momentum supplied by external forces: pressure difference : $-h\, dx \frac{dp}{dx}$.

also some skin friction : $-\tau\, dx$

from Bernoulli eq^n : $p = \text{const.} - \frac{1}{2}\rho u_1^2$. also p changes very slightly in boundary layer. $\therefore$ external force due to pressure becomes : $h\, dx\, \rho u_1 \frac{du_1}{dx}$.

$$\therefore\ -\tau + h\rho u_1 \frac{du_1}{dx} = \rho\left[\int dy \frac{\partial u}{\partial t} + \frac{d}{dx}\int u^2 dy + \underbrace{v u_1}_{\downarrow\ -\int \frac{\partial u}{\partial x} dy}\right].$$

Correction : can't use Bernoulli Th. as used above, since previously assumed not stationary flow : $\therefore$ use eq^n of motion :

$$\frac{dp}{dy} = -\rho\left(\frac{\partial u_1}{\partial t} + u_1 \frac{du_1}{dx}\right)$$

$$\therefore\ -\frac{1}{\rho} + h\left(u_1 \frac{du_1}{dx} + \frac{\partial u_1}{\partial t}\right) = \int_0^h \left(\frac{du}{dt} + \frac{d}{dx}u^2 - u\frac{d}{dx}u\right) dy$$

$$\therefore\ \frac{\tau}{\rho} = \frac{\partial}{\partial t}\int_0^h (u_1 - u)\, dy + \frac{d}{dx}\int_0^h (u_1^2 - u^2)\, dy - u_1 \frac{d}{dx}\int_0^h (u_1 - u)\, dy$$

Stationary flow : ∂t integral $= 0$

$$\therefore\ \frac{\tau}{\rho} = \frac{1}{\rho}\frac{dM}{dx} = \frac{d}{dx}\int (u_1^2 - u^2) - u_1 \frac{d}{dx}\int (u_1 - u)\, dy$$

(for flat plate) $M = \rho \int u(u_1 - u)\, dy$

define

momentum loss

$$= \rho\left[\int (u_1^2 - u^2) - \int u_1 (u_1 - u)\, dy\right]$$

$$\frac{1}{\rho}\frac{dM}{dx} = \frac{d}{dx}\int (u_1^2 - u^2) - u_1 \frac{d}{dx}\int (u_1 - u) - \frac{du_1}{dx}\int u_1 - u$$

$$= \frac{\tau}{\rho} - \frac{du_1}{dx}\underbrace{\int_0^h (u_1 - u)\, dy}_{= u_1 \delta^*}$$

$$\therefore \quad \tau = \frac{dM}{dx} + \rho \delta^* u_1 \frac{du_1}{dx}$$

$$= \frac{dM}{dx} - \delta^* \frac{dp}{dx}.$$

Now assume: $u = ay + by^2 + cy^3 + dy^4$

eq. of motion: $\eta \frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} + \rho\left(\underbrace{\frac{\partial u}{\partial t}}_{=0,\ \text{stat. flow.}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 0$ at surface

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} = \frac{1}{\eta}\frac{dp}{dx} = -\frac{1}{\eta}\rho u_1 \frac{du_1}{dx}$$

$$\therefore \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_0 = b = -\frac{u_1}{\mu}\frac{du_1}{dx}$$

$$\tau = \eta \frac{\partial u}{\partial y} = \eta a$$

require $u(\delta) = u_1$

$u'(\delta) = 0$

$u''(\delta) = 0$

Define $\lambda = \frac{\delta^2}{\mu}\frac{du_1}{dx}$

then $a = \frac{u_1}{6\delta}(12 + \lambda)$, $b = -u_1 \frac{\lambda}{2\delta^2}$ $c = -\frac{u_1}{2\delta^3}(4 - \lambda)$

$d = \frac{u_1}{6\delta^4}(6 - \lambda)$.

putting these in integrals above,

get finally a D.E. for δ.

$$\frac{dz}{dx} = \frac{0.8}{u_1}\left(\frac{-9072 + 1670.4\lambda - \lambda^2\left(47.4 + 4.8\,\frac{u_1 u_1''}{u_1'^2}\right) - \lambda^3\left(1 + \frac{u_1 u_1''}{u_1'^2}\right)}{(\lambda - 12)(\lambda + 17.76)}\right).$$

$$z = \frac{\delta^2}{\mu} = \frac{\lambda}{\left(\frac{du_1}{dx}\right)}. \qquad = f(z, x).$$

f is known as soon as we know

u_1 - i.e. pot. flow, solution.

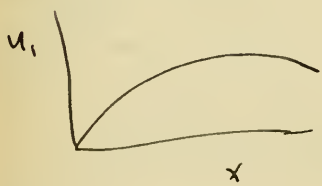
at $u_1 = 0$, forward stag. pt., $\frac{dz}{dx}$ must not blow up, ∴

equate numerator to zero & get $\lambda = 7.052$

April 10 1942

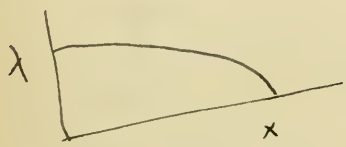
~~For small x, we always found before, $\delta \sim \sqrt{x}$, i.e. $\delta^2 \sim x$~~

integrate above D.E. subject to $\lambda = 7.052$ at $x = 0$.



since s^2 & μ are both > 0, λ has same sign as $\frac{du_1}{dx}$;

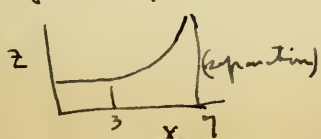
$\lambda = 0$ when $\frac{du_1}{dx} = 0$, i.e. when u_1 is a max., i.e. when pressure is min.



$\frac{du}{dy} \sim a = \frac{u_1}{6\delta}(12 + \lambda) = 0$ at $\lambda = -12$

∴ separation pt. at $\lambda = -12$

Integration gives:



cylinder: $u_1 = 7.15x - .045x^3 - .00033x^5$

separation pt. agrees within 1% of result from exact calculation —

Новая
ПРОПСТ

suppose we want $\lambda = -\bar{\lambda}$

then $z = -\frac{\bar{\lambda}}{u_1}$

$z' = \frac{\bar{\lambda} u_1''}{u_1'^2}$

from D.E., get $\frac{\bar{\lambda} u_1 u_1''}{u_1'^2} = F\left(\bar{\lambda}, \frac{u_1 u_1''}{u_1'^2}\right)$.

$\therefore$ ~~$\bar{\lambda} = F\left(\frac{u_1 u_1''}{u_1'^2}\right)$~~.

$\therefore \frac{u_1 u_1''}{u_1^2} = G(\bar{\lambda})$.

$\bar{\lambda}$	$\frac{u_1 u_1''}{u_1'^2}$
-12	∞
10	11.13
12	11.09

if λ is critical, then $\frac{u_1 u_1''}{u_1'^2} = \text{crit.} = \sigma$.

$\therefore \frac{u_1''}{u_1'} = \sigma \frac{u_1'}{u_1}$

$\therefore u_1' = C_1 u_1^{\sigma}$.

$u_1'' = C\sigma u_1^{\sigma-1} u_1'$

$u_1' = \sigma \frac{u_1'}{u_1}$

$\therefore \frac{u_1'}{u_1^{\sigma}} = C_1$

$\therefore u_1^{-\sigma+1} = (-\sigma+1) \cdot C_1 x + C_2$

if $\sigma \sim 11$, $u_1^{-10} = 10 C_1 x + C_2$

$u_1 = \frac{u_0}{(1 + 10 C x)^{\frac{1}{10}}}$.

Can prevent separation only if pressure change is very slow:

increase velocity is slow; hence decrease of pressure slow.

April 13

$$n_1 = \frac{u_0}{(1+10Cx)^{0.1}}$$

$$\therefore n_1' = -1\frac{u_0 \cdot 10C}{(1+10Cx)^{1.1}} = \frac{-u_0 C}{(1+10Cx)^{1.1}}$$

$$\therefore \frac{\delta^2}{\mu} = \frac{\lambda}{n_1'} = \frac{-10}{-\frac{u_0 C}{(\quad)}} = \frac{10}{u_0 C}(1+10Cx)^{1.1}$$

$$\therefore \delta = \delta_0 (1+10Cx)^{.55}$$

Lot of crap to show how long a stream lined body should be, to get a

lots more crap on various ways to help streamlining —

We have been ~~talking~~ always about laminar flow; this only really happens at low velocities. At higher velocities, get turbulence, which is a Bad Thing. Practically: need more pressure to get a given amount of fluid thru a pipe. Theoretically: because nothing is known about it. A given tube of stuff (say red coloring) doesn't stay together, but breaks up; And this can be a Good Thing, because it turns up to boundary layer, & helps prevent separation effect. For airplanes, have turbulent flow from a few cm. beyond edge of wing.

Reynolds Turbulence starts at same value of R (Reynolds found this with expts on lots of pipes, with different fluids & velocities

Turbulence starts at $R \sim 3000$ if $R = \frac{\bar{u} d}{\mu}$ $\bar{u}$ av. vel. d diam of tube.

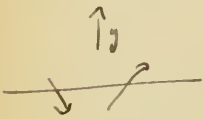
if water: $\mu = .01$

if $d = 1$ cm, $\bar{u} \sim \frac{3000 \cdot .01}{1} \approx 30$; moderate velocity

if mean velocities are $\bar{u} \neq 0$ $\bar{v} = 0$ $\bar{w} = 0$, $\bar{u}(y)$

there is no net transfer in y + z directions, only in x;

But $\overset{u}{\underset{w}{v}} = \bar{v} + v'$ if it happens that particles crossing up have average $\overline{u'} > 0$, crossing down have $\overline{u'} < 0$, e.g.



this amounts to a transport of momentum across the boundary per unit time, which gives force.

mass transport is $\rho v'$; x-momentum transport is $(\rho v')u'$;

average is $\overline{\rho v' u'}$. i.e. $\tau_{xy} = -\overline{\rho u' v'} = \rho^2 \frac{d\bar{u}}{dy}$

~~which is > 0~~

viscous case, we had $\tau_{xy} = \eta \frac{\partial u}{\partial y}$.

here we have a variable η, depends on velocity distribution in its neighborhood.

Introduce: $v^* = \sqrt{|\overline{u'v'}|} = \sqrt{\frac{\tau}{\rho}}$

We will study experiments on turbulent flow in pipes, + deduce therefrom some theory - (turbulent B.L. on flat plate)

April 15

To study ~~turbulent~~ flow, must go back to data — We found before, $\tau = \frac{p_1 - p_2}{L} \cdot \frac{y}{2}$ for any shear in cylindrical case — (y = dist. from center, $p_1 - p_2$ = pressure diff. betw. ends of pipe.) this is an eq'n of equilibrium — says nothing about origin of forces, or relation between them. Before, we assumed the viscous relation $\tau = \eta \frac{\partial u}{\partial y}$ which led to Poiseuille's formula.

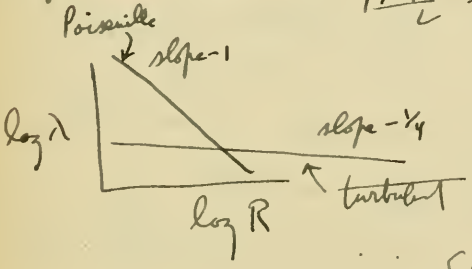
For ~~turbulent~~ case, we will take experimental relation between pressure gradient & average velocity.

~~Dimensional~~ ~~gives~~ We may write $\frac{p_1 - p_2}{L} = \lambda \cdot \frac{\frac{1}{2}\rho \bar{u}^2}{d}$ so that λ is dimensionless

Observations give $\lambda = 0.316\, R^{-\frac{1}{4}}$ $\quad R = \frac{\bar{u} d}{\mu}$.

If Poiseuille's law is written in same form, it gives:

$$\frac{p_1 - p_2}{L} = \frac{8\eta}{r^2} \bar{u} = \left(\frac{64}{R}\right) \cdot \frac{\frac{1}{2}\rho \bar{u}^2}{d}.$$



transition between laminar & ~~turbulent~~ flow happens at about $R = 2000$, which is a little beyond where they cross

[$R = 2{,}000$, $\lambda_p = .032$ $\quad \lambda_{tub.} = .045$]

With careful expts, can make laminar flow persist, even up to $R \sim 30{,}000$; then need much lower (10 times so) pressures to maintain same flow.

Let us calculate v_* at $y = r$:

$$\bar{\tau}_{(\text{near wall})} = \frac{p_1 - p_2}{L} \cdot \frac{1}{2}$$

$$= \frac{1}{2} \rho \bar{u}^2 \cdot \frac{\lambda}{d} \cdot \frac{1}{2}$$

$$= \frac{\lambda}{8} \cdot \rho \bar{u}^2 = \rho v_*^2 \quad \text{by definition}$$

$$\therefore v_*^2 = \frac{\lambda}{8} \bar{u}^2 .$$

$$= \frac{.316}{8} \left(\frac{\bar{u} d}{\mu} \right)^{-1/4} \bar{u}^2$$

$$= \frac{.316}{8} \left(\frac{d}{\mu} \right)^{-1/4} \bar{u}^{7/4} .$$

$$\frac{\bar{u}}{v_*} = 7 \left(\frac{v_* r_2}{\mu} \right)^{1/7} .$$

Can also find that in middle of tube:

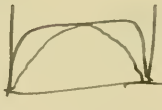
vel. in middle: $u_1 = 5/4 \, \bar{u}$

$$\therefore \frac{u_1}{v_*} = 8.74 \left(\frac{v_* r}{\mu} \right)^{1/7} .$$

Assume that in general, at a distance y from wall, $\frac{u(y)}{v_*} = f\left(\frac{v_* y}{\mu} \right)$

$u(y)$ = mean velocity at y

In fact, assume same relation as found $\nearrow$: $\frac{u}{v_*} = 8.74 \left(\frac{v_* r}{\mu} \right)^{1/7}$.

this gives velocity distrib:  instead of parabola

Now this is OK., since velocity changes only near wall, where τ is constant, & hence v_* nearly constant.

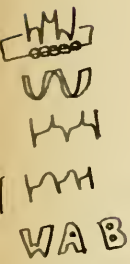
this $1/7$ power law has been misfitted over to middle of tube, (why, nobody knows). If $u = \left(\frac{r - y}{r} \right)^{1/7} u_1$

then $\bar{u} = \frac{\int y \, dy \left(\frac{r - y}{r} \right)^{1/7} u_1}{r^2/2} \sim \frac{4}{5} u_1$

this holds up to $R \sim 100{,}000$.

For larger R, exponent becomes smaller.

in general $\frac{y}{v_*} = f\left(\frac{v_* y}{\mu}\right)$ Nikuradse



in general $\frac{y}{v_*} \sim 10$ $\quad \frac{v_* y}{\mu} \sim 10$

30 $\quad$ " $\sim 10^5$

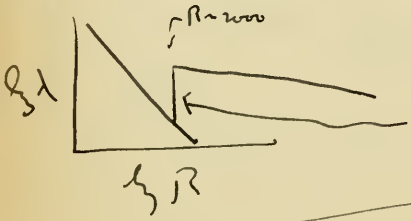
so random velocity $\sim \frac{1}{10}$ mean velocity.

$$\lambda = \frac{0.24}{\left(\log \frac{R\sqrt{\lambda}}{2.4}\right)^2}$$

for $R \sim 10^{2.5}$, $\lambda \sim .01$; doesn't change much.

For large flow, pressure gradient $\sim v^2$, low velocity, $\sim v$.

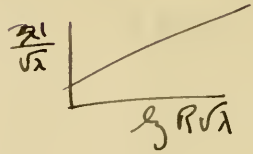
April 17 Correction: the two lines on $\log R$, $\log R$ graph should not have been connected; should not have smoothed out Blasius law to meet Poiseuille.



discontinuous change from laminar to turbulent flow.

From experiments: $\frac{u}{v_*} = 2.5 \ln\left(\frac{v_* y}{\mu}\right) + 5.5$

$\lambda = \frac{0.24}{\left[\log_{10}\left(\frac{R\sqrt{\lambda}}{2.9}\right)\right]^2}$ (Nikuradse)



$v_* = \sqrt{\frac{\tau}{\rho}}$ wall

$R = \frac{\bar{u} d}{\mu}$

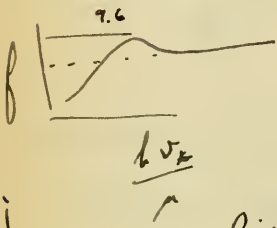
for large velocities, get $\frac{u}{v_*} = 2.5 \ln y/h + 8.5$

$h \sim$ roughness of wall —

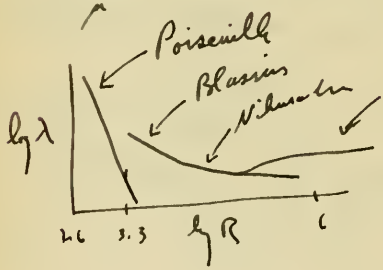
intermediate: $\frac{u}{v_*} = 2.5 \ln y/h + f\left(\frac{h v_*}{\mu}\right)$

$f = 8.5$ for large $\frac{h v_*}{\mu}$,

$f = 5.5 + \ln \frac{h v_*}{\mu}$ for small:

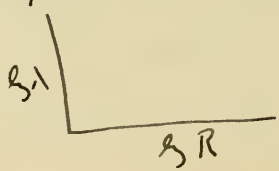


"Roughness function"



roughness of wall effect — (constant asymptotic value, higher for rougher wall.

Expt. curves:



Flow along plate.

potential flow u_1 beyond B.L. transition between surface + pot. flow will be described by formulas similar to those for tubes.

Blasius law: $\frac{u}{v_*} = 8.74 \left(\frac{v_* y}{\mu}\right)^{1/7}$

Also in Karman method; Blasius law correct in B.L. $y < \delta$.

$u = u_1$ $y > \delta$.

Momentum loss: $M = \rho \int_0^\delta u(u_1 - u)\, dy$

Write Blasius law as: $\frac{u}{u_1} = \left(\frac{y}{\delta}\right)^{1/7}$ which joins smoothly to $u = u_1$ at δ.

$\left(\frac{y}{\delta} = \xi\right)$

$$M = \rho \int_0^\delta u_1 \xi^{1/7}\left(u_1 - u_1 \xi^{1/7}\right) \delta\, d\xi$$

$$= \rho \delta u_1^2 \int_0^1 \left(\xi^{1/7} - \xi^{2/7}\right) d\xi = \rho \delta u_1^2 \left[\tfrac{7}{8}\xi^{8/7} - \tfrac{7}{9}\xi^{9/7}\right]_0^1$$

$$= \rho \delta u_1^2 \left[\tfrac{7}{8} - \tfrac{7}{9}\right] = \frac{7}{72} \rho \delta u_1^2$$

$$\therefore \tau = \frac{dM}{dx} = \frac{7}{72} \rho u_1^2 \frac{d\delta}{dx}$$

$$= .0225 \rho u_1^{7/4} \left(\frac{\mu}{\delta}\right)^{1/4}$$

$$\therefore \delta = .37 \left(\frac{\mu}{u_1}\right)^{1/5} x^{4/5}$$

$$\delta/x = .37 R^{-1/5} \qquad R = \frac{u_1 x}{\mu}$$

Laminar: $\delta/x = 4 R^{-1/2}$.

Effect turbulent flow when $4R^{1/2} = R\frac{\delta}{x} = \frac{u_1 \delta}{\nu} \sim 2{,}000$

$R \sim 250{,}000$

Exp: $R \sim 500{,}000$ for transition to turbulence.

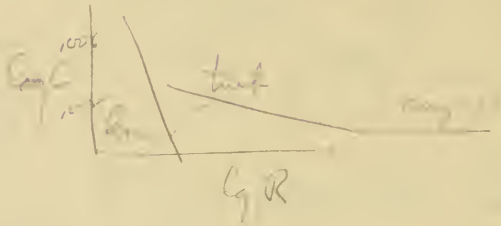
Total drag = mom. loss $= M = .036 \rho u_1^2 \times R^{-1/5} = C_D \tfrac{1}{2} \rho u_1^2 \cdot x$ where $C_D = .072$ laminar $C_D = 1.33$

April 20 apply tube results to flat plate flow.

$C = .07\ R^{-1/5}$ $R > 500$, no turb. discontinuous

$1.33\ R^{-1/2}$ $R < 500$, no laminar, change

[illegible]



Large R $R > 10^7$

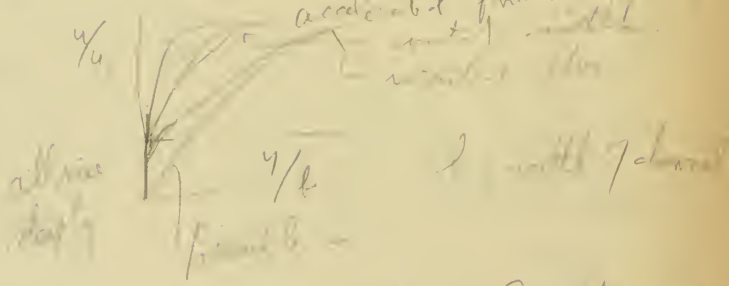
$$C = \frac{.47}{(\log R)^{2.58}}$$

[illegible]

[illegible] accelerated or [illegible] flow.

[illegible] Different [illegible] profile of [illegible]

[illegible]



[illegible]

Measure velocity profile in time of [illegible] $\frac{\delta}{\rho a_1^2}\frac{dp}{dx} = \omega$

[illegible] flow, $\frac{dp}{dx} = \omega \frac{dx}{d\pi}$, $\omega = \frac{[illegible]}{\omega_1}$

April 22 '42 | potential flow velocity $u_1 = \frac{u_0}{(1+\frac{x}{x_0})^b}$.

$b \sim 5$ from experiments. (~~turbulent flow~~).

($b \sim 10$ for laminar flow)

$z = \delta^{**}\left(\frac{u_1 \delta^{**}}{\mu}\right)^{1/4}$; $\Gamma_1 = z\frac{u_1'}{u_1}$.

above equation obtained by requiring Γ_1 be constant such that no separation occurs. ; Separation occurs for .07, ∴ take $\Gamma \sim .05$.

[laminar flow: we had $z = \lambda / u_1'$; took $\lambda \sim -10$].

$$u_1'(x=0) = -\frac{u_0}{b x_0} = \frac{\Gamma_1 u_0}{z}.$$

$$z = \frac{a}{u_1^b}\left(\int_1 u_1^b dx + c\right)$$ if take integral from front

stagnation pt., $c = 0$ $\quad$ ∴ $z_0 = \frac{a}{u_1^b}\int_{stag.pt.}^{0} u_1^b dx$

$$\therefore x_0 = -\frac{z_0}{b\Gamma_1} = -\frac{a\int^0 u_1^b dx}{u_1^b \cdot b\Gamma_1}$$

but $\frac{\int_A^0 u_1^b dx}{u_1^b}$ is of order of arc length from to separation pt. in fact smaller in about ratio $\frac{1}{b}$.

[For laminar flow, we had: $x_0 = \frac{u_0}{u_1'(stag)}\frac{K\lambda_0}{-\lambda}\cdot\frac{1}{b}$?]

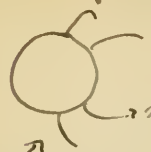
$\frac{u_0}{u_1'}$ → $\frac{1}{2}$ arc ; $\frac{K\lambda_0}{-\lambda}$ → 2 ; b → 10

$\therefore x_0 \sim \frac{1}{10}$ arc.

Turbulent case: $-\frac{a}{\Gamma_1} \sim \frac{1}{3}$; $x_0 \sim \frac{1}{100}$ arc.

therefore can make body shorter without getting separation

At high reynolds numbers



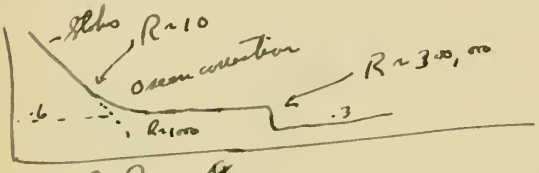
Resistance is smaller the further back the separation takes place; Hence resistance decreases (discontinuously) when turbulence sets in, because of shift of separation pt.

Drag coefficient of sphere as fn of velocity:

$$\text{Resistance} = C_D \left(\tfrac{1}{2}\rho u^2\right)(\pi r^2)$$

Stokes: $C_D \sim \frac{1}{u}$ & $C_D \sim -\log R$

Eiffel



Flow in wake can be investigated; Wake spreads as $\sqrt{}$ distance from body in 2-dim. problem; as $\sqrt[3]{}$ in a body of revolution

Back to potential flow

We know it isn't good, but after having discussed the real flow, can judge how good it is. Can be solved much easier.

Lift of an airfoil:

Impose condition that fluid can't flow around sharp edge.
(We found separation even for moderate curvatures).
With good streamlining, no separation until end;
[~~wake~~ turbulence] hence potential flow solution is useful

Velocity potential φ can be introduced:

$\underline{v} = \nabla\varphi$ O.K. if $\nabla\times\underline{v} = 0$ everywhere
no vortices.

O.K. for laminar flow; not too bad even for turbulent flow, except in B.L.

Bernoulli eq'n:

replace ~~$(\underline{u}\cdot\nabla)\underline{u}$ by u~~

$\underline{v}\cdot\nabla u$ by $v\cdot\frac{\partial \underline{v}}{\partial x}$ in Euler eq'n

& get $p = p_0 - \frac{1}{2}\rho\underline{v}^2$.

Boundary Condition: $v_{normal} = 0$ [can't satisfy $v_{tan} = 0$]

i.e. $\frac{\partial\varphi}{\partial n} = 0$.

Continuity: $\nabla\cdot\underline{v} = 0$ whence $\nabla^2\varphi = 0$.

So need only solve Laplace's eq'n subject to $\frac{\partial\varphi}{\partial n} = 0$ on surfaces, & v_∞ = const. given.

Soluble for some 3-dim. cases. (e.g. sphere).
Most results in 2-dim. Good approx., for airfoils.

---

if $\varphi = \varphi(x+iy)$ then $\nabla^2\varphi = 0$.

Or: $\omega(x+iy) = \varphi + i\psi$; $\nabla^2\varphi = 0$ and $\nabla^2\psi = 0$.

and $\underbrace{\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}}_{=u}$, $\underbrace{\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}}_{=v}$ (Cauchy Riemann)

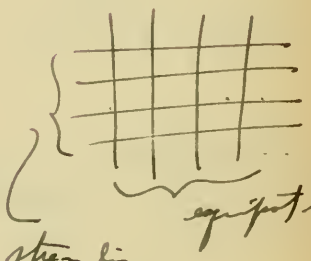
ψ = constant on a streamline — (stream function).

$$\left[\ \frac{dy}{dx} = \frac{v}{u} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y}\right.$$

Consider: $\omega = z$.

then $\varphi = x$ $\psi = y$:

$u = 1$
$v = 0$

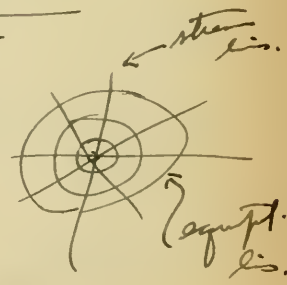


$\omega = \Gamma \log z$ $z = re^{i\alpha}$

$\varphi = \Gamma \log r$ $\psi = \Gamma\alpha$

$v_r = \frac{\partial\varphi}{\partial r} = \Gamma/r$.

source at center of strength Γ.



"Yield" of source $= \int_0^{2\pi} r\,\Gamma/r\,d\theta$

$= 2\pi\Gamma$.

If $\Gamma = i\,\Gamma'$, $\varphi = -\Gamma'\alpha$, $\psi = \Gamma' \log r$

$v_r = \frac{\partial \varphi}{r\partial \alpha} = -\frac{\Gamma'}{r}$

stream lines around circle, equipot. radial.

define Circulation $= \oint \underline{v} \cdot d\underline{s}$ around streamline

$$= \int_0^{2\pi} -\frac{1}{r}\Gamma' \cdot r\, d\theta = -2\pi\Gamma'$$

to any potential flow, (e.g. for cylinder), can add a circulation, without changing Boundary condition: it will still leave flow normal to body, & will still have prescribed value at ∞.

i.e. solution in 2 dim. is not unique: therefore must add a condition – which we did: no flow around a trailing edge.

April 2442 $w = \varphi + i\psi$; $u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}$, $v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$.

ψ = constant: streamlines. φ = constant: equipot. lines.

Also: $\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv$ "complex velocity".

$-i\frac{\partial w}{\partial y} = -i\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x} = u - iv$

If $w = z e^{i\alpha}$, $\frac{dw}{dz} = e^{i\alpha}$ = flow at angle α with x-axis.

$w = \frac{\Gamma}{2\pi i} \log z$ Γ real: source of strength Γ.

Γ imaging: circulation Γ'.

$(\Gamma = -i\,\Gamma')$

Flow around cylinder: cylinder surface a streamline.

Suppose v_∞ = const. in x direction.

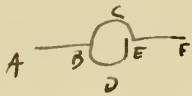
one streamline is:

Required function is $w = z + z^{-1}$.

if $z > 1$, real, $w > 0$, real;

$z < -1$, real, $w < 0$, real;

on unit circle $z = e^{i\beta}$, $w = 2\cos\beta$, real;

$\therefore$ 

in z-plane = real axis in w-plane.

B: $z = -1$, $w = -2$ E, $z = 1$, $w = 2$ C, $z = i$, $w = 0$ D $z = -i$, $w = 0$

Can add now a circulation $-\frac{i\Gamma}{2\pi}\log z$ + still have a solution.

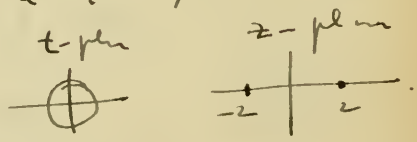
Most important application is to theory of airfoils.

$z = z(t)$; $w = t + \frac{1}{t}$.

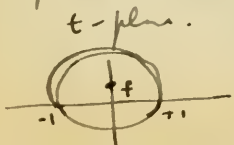
$z(t)$ transforms circle into some other contour. (e.g. airfoil).

$z = t + \frac{1}{t}$ transforms unit circle in t-plane to

str. line $-2 < z < 2$ in z-plane



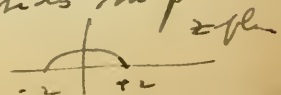
Displace circle a little:



What happens in z-plane?

Turns out that both sides map

into a circular arc:



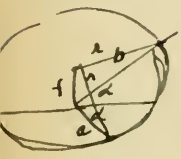
top of circle: $t = \left(\sqrt{1+f^2} + f\right) i$

$$z = (\quad) i - \frac{i}{(\quad)} = i\left[\frac{(\quad)^2 - 1}{(\quad)}\right]$$

$$= i\left[\frac{1+f^2+2f\sqrt{\quad}+f^2-1}{\sqrt{\quad}+f}\right.$$

$$= 2if\,\frac{f+\sqrt{\quad}}{\sqrt{\quad}+f} = 2if \;.$$

To show upper + lower part of circle go into same arc:



$$R^2 = f^2 + a^2 - 2af\cos(90°+\alpha)$$
$$= f^2 + a^2 + 2af\sin\alpha$$

also $R^2 = f^2 + b^2 - 2bf\sin\alpha$

$$R^2 - f^2 = 1 = a^2 + 2af\sin\alpha$$
$$= b^2 - 2bf\sin\alpha$$

$c = f\sin\alpha$

whence $a = -c + \sqrt{1+c^2}$

$b = c + \sqrt{1+c^2}$

where $ab = 1$

a: $t = ae^{i\alpha}$

b: $t' = be^{i\alpha} = \frac{1}{a}e^{-i\alpha} = \frac{1}{t}$.

and $t + \frac{1}{t} = t' + \frac{1}{t'}$. Q.E.D.

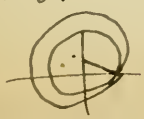
z the same for a + b.

Turns out that z-plane arc is circular. & till not

Consider a circle tangent to above

an airplane wing.

circle at $t = +1$, but larger radius:



This gives:

Kutta.

Zukowski

almost a wing!

Can get diffent shapes by taking different values of f & radius of circle. "Kutta-Joukowski profiles"

Take case of circular arc: get from circle with centre at if, radius $\sqrt{1+f^2}$. Flow around this circle:

$$w = \left[(t-if) + \frac{1+f^2}{t-if}\right] \leftarrow \text{to take care of edge of radius}$$

$$u_\infty \qquad -\frac{i\Gamma}{2\pi}\log(t-if)$$

Γ is undetermined;

$$\frac{dw}{dz} = \frac{dw}{dt}\frac{dt}{dz} = \frac{\frac{dw}{dt}}{\left(1-\frac{1}{t^2}\right)}$$

velocity $= \infty$ at $t = \pm 1$, unless $\frac{dw}{dt} = 0$ there.

this would mean a flow around the trailing edge which cannot be. ∴ adjust w so that $\frac{dw}{dt} = 0$ at $t = \pm 1$

[for wing of finite thickness $t=-1$ is not on wing - no difficulty]

[velocity at leading edge may be quite large for thin wing]

This condition fixes Γ. $\quad \frac{dw}{dt} = u_\infty\left(1 - \frac{1+f^2}{(t-if)^2}\right) - \frac{i\Gamma}{2\pi(t-if)}$

at $t=1$: $\quad u_\infty\left(1 - \frac{1+f^2}{(1-if)^2}\right) - \frac{i\Gamma}{2\pi(1-if)} = 0$

$$\frac{i\Gamma}{2\pi} = -2\pi i(1-if)\left(\frac{1+f^2-2if-1-f^2}{(1-if)^2}\right)$$

$$= -2\pi i\,\frac{-2f^2-2if}{1-if}$$

$$= +4\pi if\,\frac{f+i}{1-if}$$

$$= -4\pi f\,u_\infty$$

Suppose u_∞ makes angle α with x-axis: Simply replace ~~$u_\infty \log [(t-if)e^{-i\alpha} + 1+f^2]$~~ $(t-if)$ by $(t-if)e^{-i\alpha}$. (rotate thro $-\alpha$).

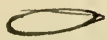
If solve for Γ this way, get

$\Gamma = -4\pi(\sin\alpha + f\cos\alpha)v_\infty$

$\therefore$ circulation is clockwise.

this means that velocity is greater above than below; hence get a force up.

[in fact, in general get no force without a circulation].

For shape:  get no lift for $\alpha = 0$.

But for shape → get lift even for $\alpha = 0$; [usually adjusted to be = weight of plane at cruising speed]

April 27'42

If a is length of wing, in general:

$\Gamma = -a\pi(f\cos\alpha + \sin\alpha)|v_\infty|$. ($a$ instead of 4 is only difference)

(a = width of wing)

Potential flow, 2 dim:

$$\underline{F} = -l\int p\, d\underline{s}$$

$$F_y + iF_x = -l\left[\int [p\cos(n,y) + ip\cos(n,x)]\, ds\right.$$

or $F_y + iF_x$ per unit length $= \int p(dx - i\,dy)$

but $p = p_0 - \frac{1}{2}\rho v^2$

$\cos(n,y) = -\frac{dx}{ds}$

$\cos(n,x) = +\frac{dy}{ds}$

$$\therefore F_y + iF_x = -\frac{1}{2}\rho\int |v|^2(dx - i\,dy)$$

but $|v| = \frac{dw}{dz}$

$$|v|^2 = \frac{dw}{dz}\left(\frac{dw}{dz}\right)^* =$$

$$\therefore \bar{F}_y + i\bar{F}_x = -\tfrac{1}{2}\rho \oint \frac{dw}{dz}\,\frac{dw^*}{dz^*}\,dz^*$$

$$= -\tfrac{1}{2}\rho \oint \frac{dw}{dz}\,dw^*$$

$$w = \varphi + i\psi$$

Since contour of body is a streamline, i.e. ψ = const. on surface of body. $\therefore$ dw is real $= d\varphi$

Whence $dw^* = dw$ and

$$\bar{F}_y + i\bar{F}_x = -\tfrac{1}{2}\rho \oint \frac{dw}{dz}\,dw$$

(Blasius)

$$= -\tfrac{1}{2}\rho \oint \left(\frac{dw}{dz}\right)^2 dz$$

w at inf. $= |v_\infty| e^{-i\alpha} z$ + lower powers of z.

$$\left[\frac{dw}{dz} = |v_\infty| e^{-i\alpha} = u - iv\right.$$

Body can be put around origin; then expand w in Laurent series around origin; $w = c_0 z + c_1 \log z + \sum \frac{c_{n+1}}{z_n}$

$$\frac{dw}{dz} = c_0 + \frac{c_1}{z_1} + \sum \frac{c_n}{z_n} \cdots$$

Hence $\oint \left(\frac{dw}{dz}\right)^2 dz = 2\pi i \left[\operatorname{Res}\left(\frac{dw}{dz}\right)^2 \text{ at } z=0\right] = 2\pi i \cdot 2c_0c_1$

$$= 4\pi i\, c_0 c_1$$

But c_1 term is circulation term — in fact

$$c_1 = -\frac{i\Gamma}{2\pi} \quad \text{and} \quad c_0 = |v_\infty| e^{-i\alpha} = \{u_\infty\} - i\{v_\infty\}$$

$$\therefore -\tfrac{1}{2}\rho\oint = \cancel{4\pi} i \cdot (u_\infty - i v_\infty)\left(\frac{-i\Gamma}{\cancel{2\pi}}\right) \cdot -\rho/2$$

$$= -\rho\Gamma(u_\infty - i v_\infty)$$

$$\therefore \boxed{\begin{aligned} F_x &= -\rho\Gamma u_\infty \\ F_y &= \rho\Gamma v_\infty \end{aligned}}$$

i.e. if $\Gamma = 0$, no force! Total force $\sim |v_\infty|$, direction $\perp$ to it. $[\underline{F} \cdot \underline{v} = 0]$. Hence doesn't do any work, no resistance associated with it. $|\text{Lift}| = \rho\Gamma|v_\infty|$

if v_∞ is $+x$ direction $\longrightarrow$, then $F_y \uparrow$ (since (Γ is negative). Put Γ in:

$$F_y = \rho\pi a |v_\infty|^2 (\text{fcos}\,\alpha + \sin\alpha)$$

Numbers: ρ (air) $\sim 10^{-3}$ gm/cm^3 $v \sim 500$ km/hr ~ 150 m/sec

$v^2 \sim 2 \cdot 10^8$ (cm/sec)^2.

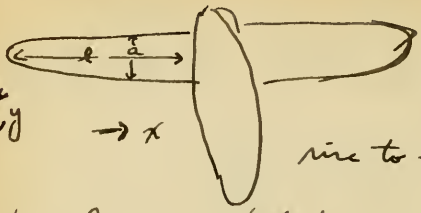
$a \sim 5$ m $= 5 \cdot 10^2$ cm.

Then lift is $10^{-3} \cdot \pi \cdot 5 \cdot 10^2 \cdot 2 \cdot 10^8$ (fcos α ...) dyn

$\cancel{8} \cdot 10^8$ ( ) dyn/cm.

100 ( ) kgwt./cm

if $l = 10$ m. ~ 100 tons.



End effects give rise to a velocity in x-direction. gives rise to end drag. Also get often a profile drag (which is smaller).

Still have pot. flow, $\frac{du}{dy} = \frac{dv}{dx}$.

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \frac{\partial u}{\partial y} = \frac{1}{v}\frac{du}{dt}$$

u at trailing edge $u_t = \int_{\text{front}}^{\text{back}} \frac{\partial v}{\partial x}\, dy$

$$u_{t\,\text{upper}} - u_{t\,\text{lower}} = \frac{\partial}{\partial x}\int_{\text{upper}} v\, dy - \frac{\partial}{\partial x}\int_{\text{lower}} v\, dy$$

$$= \frac{\partial}{\partial x}\,(\text{circulation}(x))$$

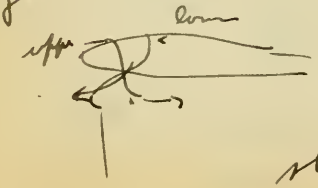
$$= \frac{\partial(|\Gamma(x)|)}{\partial x}.$$

Behind wing, get a surface of discontinuity where velocities on upper & lower sides are in opposite directions.

$\Gamma = 0$ at end of wing, then increases to some max. value;

$\therefore \frac{\partial \Gamma}{\partial x}$ is positive. $\therefore u_{t\,\text{upper}} \rightarrow$ $u_{t\,\text{lower}} \leftarrow$.

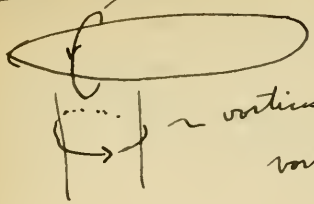
get vortices with axes in direction of motion of plane.



Behind wing have vortex lines in direction of motion (y-direction)

strength $= \frac{d\Gamma}{dx}\,a_x$ at x

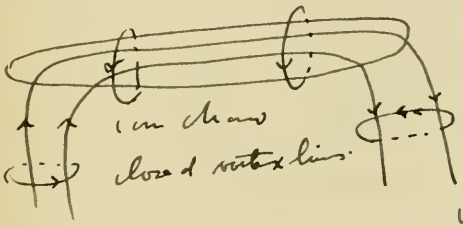
April 29 circulation around wing



~ vortices behind wing

vortex lines in direction of flight.

draw lines in direction of $\nabla \times \underline{v}$; $\nabla \cdot \nabla \times \underline{v} = 0$, hence the vortex lines never end

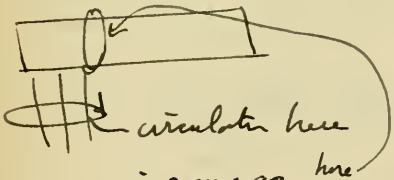


strength of vortex line : $d\Gamma = \Gamma' dx$

(in closed vortex lines.

velocity produced by one vortex filament $= \frac{d\Gamma}{2\pi r}$

$\left[d\Gamma = \oint v\,ds = 2\pi r v \right.$



circulation here is same as here

velocity produced by vortex fil. at $x = \xi$ is $\frac{d\Gamma_\xi}{2\pi(x-\xi)} = \frac{\Gamma'(\xi)\,d\xi}{2\pi(x-\xi)}$

$\therefore$ total downward vel. at $x = \int \frac{\Gamma'(\xi)\,d\xi}{2\pi(a-\xi)}$

This would be so if vortex line went straight from $+\infty$ to $-\infty$; but effectively we have only half of this, since vortex line go away at ∞. $= \omega(x)$.

$\therefore \omega(x) = \frac{1}{4\pi} \int \frac{\Gamma'(\xi)\,d\xi}{x-\xi}$. downward velocity .

Now Force is ⊥ velocity of airstream; but we now have a downward component of velocity w ↓ to v_∞ → resultant.

∴ force is ↗ instead of ↑; can resolve into two comp. ↑ lift + ↓ drag.

Ratio $\frac{\text{drag}}{\text{lift}} = \frac{w}{v_\infty}$.

but lift $= \rho v_\infty \Gamma$

$\therefore$ drag $= \rho w \Gamma$.

Total lift $= \rho v_\infty \int \Gamma \, dx$

total drag $= \rho \int w(x) \Gamma(x) \, dx$.

Want large lift, small drag, i.e. $\int \Gamma \, dx$ a max., $D = \int \Gamma w \, dx$ a min.

i.e. make distribution of Γ so that drag D is a min.

can do this by changing width of wing, or shape.

Suppose

Given by increase γ of Γ over small distance β must give same change of D wherever applied.

change of $D = \gamma \left(\beta w(x_0) + \int \Gamma \, \delta w \, dx \right)$

if change Γ by γ at x_0 but $\delta w(x) = \frac{1}{4\pi} \left[\frac{\gamma}{(x - x_0)} - \frac{\gamma}{x - (x_0 + \beta)} \right]$.

$$\int \Gamma(x)\,\delta\omega(x)\,dx = \frac{\gamma}{4\pi}\int_{-\ell}^{\ell'} \Gamma(x)\left(\frac{1}{x-x_0} - \frac{1}{x-x_0-\beta}\right)dx$$

$$= \frac{\gamma}{4\pi}\int_{-\ell}^{\ell}\left\{\frac{\Gamma(x)}{x-x_0}\,dx - \int_{-\ell-\beta}^{\ell-\beta}\frac{\Gamma(x+\beta)}{x-x_0}\,dx\right\}$$

if $\Gamma(\pm\ell)=0$: $= -\frac{\gamma}{4\pi}\int_{-\ell}^{\ell}\frac{\beta\Gamma'(x)\,dx}{x-x_0} = -\frac{\gamma\beta}{4\pi}\left(-\omega(x_0)\right)4\pi$

$$= \beta\gamma\,\omega(x_0)\,.$$

$$\therefore \delta D = 2\gamma\beta\omega(x_0)\,.$$

if this is to be the same everywhere, we want

$\omega(x_0) = \text{const}$. surface of discontinuity moves down all together with same velocity

Now want $\int \frac{\Gamma'(\xi)\,d\xi}{x-\xi}$ independent of x.

this requires an elliptic distrib. of Γ:

$$\Gamma = \sqrt{1-\left(\frac{\xi}{\ell}\right)^2}$$

[if $\ell = 1$, $\Gamma' = \frac{-\xi}{\sqrt{1-\xi^2}}$, $\int_{-1}^{1}\frac{-\xi\,d\xi}{(x-\xi)(\sqrt{1-\xi^2})}$.

$$= \int_0^{\pi}\frac{\cos\varphi\,\sin\varphi\,d\varphi}{(x-\cos\varphi)\sin\varphi} = \int_0^{\pi}\frac{\cos\varphi\,d\varphi}{x-\cos\varphi}$$

$$= \int_0^{\pi}\left(1 - \frac{x}{x-\cos\varphi}\right)d\varphi$$

$$= \pi - x\int_0^{\pi}\frac{d\varphi}{x-\cos\varphi}$$

$$= 0$$

if $|x| < 1$.

To achieve this, make a (width) wing elliptically,

can do this : two ellipses.

Better → increase in drag for rectangular shape 4%.

for 4:1 length to width ratio.

in between shapes, drag only a few % greater than for elliptic.

if $\Gamma = \Gamma_0 \sqrt{1 - x^2/\ell^2}$ $\quad w = \frac{\Gamma_0}{4\ell} = v_\infty \frac{\pi a}{4\ell} (\bar{\alpha} + \tan\alpha)$

$= v_\infty \cdot \frac{\pi}{4} \cdot \left(\frac{a}{\ell}\right) (\bar{\alpha} + \tan\alpha)$

$\left[\frac{w}{v_\infty} \sim \frac{a}{\ell} \text{ small} \right.$

$\therefore D \ll L.$

$D/L \sim a/\ell$

Total lift $\sim v_\infty^2 a \ell$

Drag $\sim v_\infty^2 a^2$

$\therefore$ Drag independent of length.

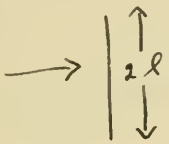
long wing desirable [physically, all because drag arises from ends of wing

May 1 1942

We found: "induced" resistance does not depend on length, only on width of wing; but lift depends on length as well (prop. to it). Resistance ~ Γ Lift ~ Γ ∴ if increase lift by tilting wing, increase resistance as square of lift.

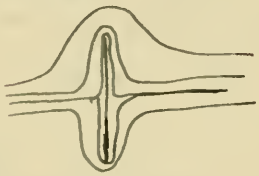
Discuss now "Profile Drag"

simple case: plate ⊥ to liquid.

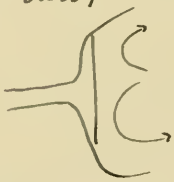


No lift – by symmetry but high resistance.

Naïve potential theory gives:

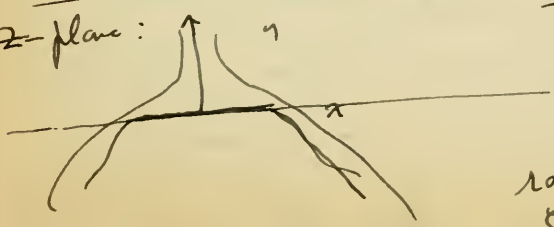


but this would give no drag – symmetrical on both sides. But we know this aint so – but will get separation at ends of plate



practically at rest here; some vortices

z-plane:



Introduce w: $\frac{dw}{dz} = u - iv$

$w = \varphi + i\psi$

Let $\zeta = \frac{dz}{dw} = \frac{u+iv}{|v|^2}$.

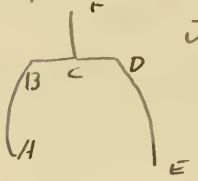
radius vector to a point in the ζ-plane has direction of velocity

in z-plane. magnitude is its reciprocal.

real axis in w-plane [$\psi = 0$] corresponds to

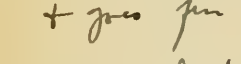
stream lines: in z-plane. Let C be at

w-plane:



origin of w-plane; − real axis coincides to F-C, + real axis to both CBA & CDE.

ζ-plane: we know velocity from C to D is in + direction, + goes from 0 to some finite + value; ∴ ζ goes from 0 to a finite value. Also C-B has vel. in − x direction along AB & DE we know magnitude of velocity by Bernoulli's theorem. If no vortices inside ABDE, then



can apply Bernoulli thus; p constant in "dead water" hence pressure constant along AB & DE; ∴ velocity the same on AB & DE as at B & D. Hence in ζ-plane get a circle:

take this as unit circle. Then problem is to map the outer + real axis in w-plane. This is done by: $\frac{1}{\sqrt{w}} = \zeta + \frac{1}{\zeta}$. if ζ is real, & > 1, then $\sqrt{w} < 1$, from 0 to 1

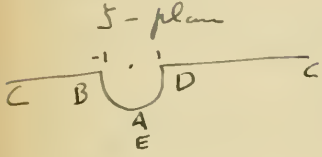
also if $\zeta < -1$, $\omega < 1$.

also if $\zeta = e^{i\alpha}$, $\frac{z}{\sqrt{\omega}} = 2\cos\alpha$, $\omega = \frac{1}{\cos^2\alpha}$,

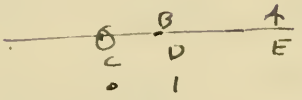
which goes from 1 to ∞. $(-\pi/2 < \alpha < 0)$.

So this works:

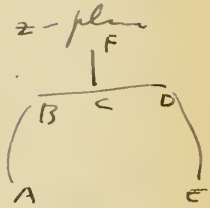
ζ-plane



ω-plane:



z-plane



$\frac{dz}{d\omega} = \zeta$ but $\frac{z}{\sqrt{\omega}} = \zeta + \frac{1}{\zeta}$;

$z = \int \zeta\, d\omega$ i.e. $\zeta = \frac{1}{\sqrt{\omega}} \pm \sqrt{\frac{1}{\omega} - 1}$

as $\frac{1}{\sqrt{\omega}} \to \infty$, $\zeta \to \infty$, $\therefore$ take + square root.

Now $\overline{CD} = l = \int_0^1 d\omega \left(\frac{1}{\sqrt{\omega}} + \sqrt{\frac{1}{\omega} - 1}\right)$

but $\int d\omega \left(\frac{1}{\sqrt{\omega}} + \sqrt{\frac{1}{\omega} - 1}\right) = 2\sqrt{\omega} + \sin^{-1}\sqrt{\omega} + \sqrt{\omega(1-\omega)}$.

$\therefore l = 2 + \sin^{-1} 1 = 2 + \pi/2$

length of plate $= 2l = 4 + \pi$.

if ω is imaginary, $z = 2\sqrt{\omega} + i \ln\left(\sqrt{\omega} + \sqrt{\omega + 1}\right) + \pi/2 - i\sqrt{\omega(\omega - 1)}$.

(can also write:

for large ω, $z \approx 2\sqrt{\omega} - i\sqrt{\omega(\omega-1)}$.

$\therefore x = 2\sqrt{\omega}$, $y = -\omega$ } which gives a parabola -

How is velocity calc'd AB, DE related to velocity at infinity?

at $\bar{r}$, $w = -\infty$, $\therefore \zeta = -i$ as it should. (i.e. directed downward, & magnitude 1. $\therefore$ velocity at surface of discontinuity $= v_\infty$ (we have really used units? vel. at infinity.

Can calculate resistance: $F = \int (p_{front} - p_{back})\, dx$.

$p_{back} = p_B$; $p(x) = -\frac{1}{2}\rho \int_{+(2+\pi/2)} \{v^2(x) - v^2(B)\}$ ← this is wrong

$= -\frac{1}{2}\rho \int_{-(2+\pi/2)} \left(\frac{1}{\zeta^2} - 1\right) d\frac{x}{2}$

$= -\frac{1}{2}\rho \int_{\neq 0}^{+\infty} \left(\frac{1}{\zeta^2} - 1\right) \zeta\, dw$

$= 2 \cdot \frac{1}{2}\rho \int_0^1 \left(\frac{1}{\zeta} - \zeta\right) dw$

$= \rho \int_0^1 2\sqrt{\frac{1}{w} - 1}\, dw = \pi\rho$

$\therefore \bar{F} = \pi\rho$ but what are the units? actually $\bar{F} = \rho \cdot v_\infty^2 \cdot \pi \cdot \left(\frac{2\ell}{4+\pi}\right)$ since force is prop. to length. Hence drag coeff is, if

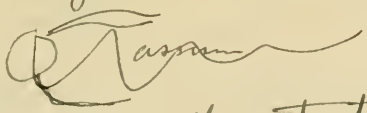
$F = \frac{1}{2}\rho v_\infty^2 \cdot 2\ell \cdot C_D$, $\quad C_D = \frac{2\pi}{4+\pi} \sim .85$.

Observed $C_D \sim 1.3$.

about as good as to be expected, in view of dead approx —

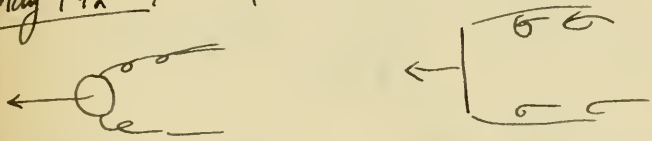
this done by Helmholtz.

Levi-Civita: in general for a body, assume separation at some point



Rather think a set of self consistent calculations would be good — assume a separation pt., calculate potential flow, then use Karmán theory to find separation pt. —

May 4'42 | Karman Vortex trail.



vortices on neighborhood of surface of discontinuity

A surface of disc. may be considered as a set of vortex lines

Vortex lines are grouped around centers; get discrete vortex centers & vortices.

small vortices would be unstable — only one stable configuration of vortices along two parallel lines (really we have parabolic lines, in case of plate)

Karmán studied the case of 11 lines.

Suppose we have vortices along a line; this is stable if they are equidistant. If one is removed

a little; vortices on right make it move up, on left, down; but it is closer to one than other, hence it moves out of line. Hence a linear set of vortices is in unstable equilibrium. Actually, always have two sets; one from each side of body.

Suppose two sets: — o — o — o — o —

of opposite direction — o — o — o — o —

lower row produces a vel. to right on upper

upper " " " " " right on lower

∴ whole thing moves to right.

No force up or down from one to next. Hence equilibrium

Now displace one of ~~lower~~ upper vortices to left. It will still move away; but not so much; upward velocity due to near upper is partly compensated by downward vel. due to lower near one. Hence still unstable.

But now space them differently: o o o o
o o o

if put them at ½-way points, can find a value of ratio $\frac{\text{dist betw vortices}}{\text{" " rows}}$ which gives equilibrium.

Handb. der Exp. Ph. 4, 1, 155

Ratio $\frac{h}{l} = .281$

h = dist. between rows
l = " " vortices.

$\left(\sinh \frac{\pi h}{l} = 1\right)$ ↑

In actual flow, vortices separate alternately from upper & lower surfaces

Turns out that $\bar{u} = \frac{\Gamma}{2l} \tanh \frac{\pi h}{l} = \frac{\Gamma}{2\sqrt{2}\, l}$.

Drag $D = L \rho \left[\frac{\Gamma}{l} (u - 2\bar{u}) h + \frac{\Gamma^2}{2\pi l} \right]$

(length, density, vel. of body; velocity of vortex trail.)

d = width of plate
L = length of plate

$D = C_D \frac{1}{2} \rho u^2 L \cdot d$

$C_D = 2 \frac{l}{d} \left(.744 \frac{\bar{u}}{u} - .314 \left(\frac{\bar{u}}{u}\right)^2 \right)$.

agrees within exp. error

Kármán & Rubach made measurements on plate & cylinder.

	plate	cylinder	theory
d =	1.75	1.5	
h =	3 cm	1.8 cm.	
l =	4.8 "	6.4 "	
h/l =	.305	.28	.281
$\bar{u}/u$ =	.20	.14	→ vortices move slowly in comparison with body —
calc. C_D =	.80	.46 → calculated	not bad.

$\frac{\bar{u}}{u}$ is independent of
h/l " " " "
drag coeff. " " "

D'Alembert's Paradox

No net force, in potential flow: $\underline{F} = -\int_{\text{surface}} p \, d\underline{\sigma}$

$$= \frac{1}{2}\rho \int v^2 d\underline{\sigma}$$

$$= \frac{1}{2}\rho \Big[\underbrace{\int v^2 d\sigma}_{\text{sphere at } \infty} - \underbrace{\int \nabla v^2 \, d\tau}_{\text{volume between surfaces.}}$$

but $\nabla v^2 = \underline{v}\, \nabla \cdot v - \underline{v} \times \nabla \times \underline{v}$

if incompressible: $\nabla \cdot v = 0$

if irrotational $\nabla \times v = 0$

$$\therefore \int d\tau \, \nabla v^2 = 0$$

but $\int_{\text{sphere at } \infty} v^2 d\underline{\sigma} = 0$ also, since

$$v = \nabla \varphi$$

$$\varphi = v_\infty \cdot x + \text{const.} \cdot \frac{a}{r} + \frac{1}{r^2} \cdot \underbrace{\ }_{\text{source.}}$$

$$\therefore v = v_\infty \underline{i} + \frac{c}{r^2} \cdot \underline{r}/r$$

$$v^2 = v_\infty^2 + \frac{c}{r^2} \cdot 2\frac{x}{r} v_\infty$$

but $v_\infty^2 = \text{const.}$, $\int \text{const.} \, d\sigma = 0$

and $c = 0$ if no sources.

$$\therefore \int v^2 d\underline{\sigma} = 0$$

and $\underline{F} = 0$.

May 6

Compressible fluids:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{v} = 0$$

or: $$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho \nabla \cdot \underline{v}$$

Eq'n. of motion still the same:

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v} = -\frac{1}{\rho} \nabla p$$ (without viscosity)

Need another equation relating p and $\rho - \rho_0$.

Can get this by consideration of energy:

Change of internal energy per unit mass:

$dE + p\,dV$ = energy received by heat conduction and viscosity.

But heat conduction is rather a slow process; also viscous forces are small & to be considered only at boundaries;

Hence $T dS = dE + p dV = 0$.

or: $\frac{dS}{dt} = 0$ entropy conserved for a small element. [Doesn't mean that entropy is constant everywhere – but that each element "carries its entropy with it"] "adiabatic"

∴ [For very slow motion, heat conduction enough to keep temperature constant "isothermal".]

For gases, for adiabatic behavior:

$$p = \alpha \rho^{\gamma} \qquad \gamma = \frac{c_p}{c_v} = 5/3 \text{ for monatomic gases}$$

$7/5$ diatomic

$\gamma \begin{matrix} > 1 \\ < 7/5 \end{matrix}$ polyatomic

If entropy is constant over whole system for all time (which happens except in shock waves)

Define: $\left(\frac{\partial p}{\partial \rho}\right)_S = c^2$

For gas: $\frac{\partial p}{\partial \rho} = \alpha \gamma \rho^{\gamma - 1}$

$= \gamma \frac{p}{\rho} = \gamma R T$.

Define compressibility $\kappa = \frac{d\rho/\rho}{dp}$

$\therefore c^2 = \frac{\kappa}{\rho}$.

Continuity eq'n. $\frac{\partial \rho}{dt} + \underline{v} \cdot \nabla p = -\rho c^2 \nabla \cdot \underline{v}$.

---

1. Small disturbances: $\underline{v}$ small, changes in ρ small. Then can neglect 2nd order terms.

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla p}{\rho}$$

$$\frac{\partial p}{\partial t} = -\rho c^2 \nabla \cdot \underline{v}$$

Now can consider $\rho = \rho_0$, since ~~this~~ one right

are already 1st order.

$$\therefore \frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial t}\left(-\rho_0 c_0^2 \nabla \cdot v\right) =$$

$$= -\rho_0 c_0^2 \nabla \cdot \frac{\partial v}{\partial t}$$

$$= -\rho_0 c_0^2 \nabla\left(-\frac{\nabla p}{\rho_0}\right)$$

$$= c_0^2 \nabla^2 p$$ wave eq'n. sound waves.

this is Acoustics.

Simple cases: plane disturbance.

$p(x,t)$.

$$p = f(x - c_0 t) + g(x + c_0 t)$$

(plane waves to right) (plane wave to left

If have piston in tube:

get waves to right:

$$p = f(x - c_0 t)$$

$$\frac{\partial p}{\partial x} = -\frac{1}{c_0}\frac{\partial p}{\partial t}$$

But $\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} = -\frac{1}{c_0 \rho_0}\cdot\frac{\partial p}{\partial t}$

$$\therefore u = \frac{p}{c_0 \rho_0}$$

$c_0 = 330$ m/sec	air
1470	" water
5000	" iron

water: $\rho_0 c_0 = 1.5 \cdot 10^5$ $\#/m^2/sec$

$$\therefore p = 1.5 \cdot 10^5 \cdot u$$

if pressure measured in bars, $p = .15\, u$.

Air: $\rho_0 c_0 = 40$; $p \sim 40u$ dyn/— — very small pressures

---

Sound: energy waves $\sim 10^{-6}$ erg/cm²/sec ; kin energy $\sim \rho u^2$ ∴ $\rho u^2 c_0 \sim 10^{-6}$

$$u^2 \sim \frac{10^{-6}}{40} \sim 2.5 \cdot 10^{-8}$$

$$u \sim 10^{-4} \text{ cm/sec.}$$

∴ $p \sim .01$ dyn/cm² ..

Spherical sound waves :

$$\nabla^2 p = \frac{1}{c_0^2} \ddot{p}$$

plane : $\frac{1}{r} \frac{\partial^2}{\partial r^2}(rp) = \frac{1}{c_0^2} \ddot{p}$

$$\frac{\partial^2}{\partial r^2}(rp) = \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2}(rp)$$

$$\therefore rp = f(r - c_0 t) + g(r + c_0 t) .$$

If u is radial velocity :

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0} \nabla p = -\frac{1}{\rho_0} \frac{\partial p}{\partial r}$$

$$= -\frac{1}{\rho_0}\left[-\frac{f+g}{r^2} + \frac{f'+g'}{r}\right]$$

$$\therefore u = \frac{1}{c_0 \rho_0} p + \frac{1}{\rho_0} \frac{1}{r^2} \int (f+g)\, dt$$

velocity has one term like plane waves, important at long dist ; but at small distances have the

"afterflow" term. Origin is because a given radial velocity means larger transport of matter at large distances than small; hence fluid at small distances must try to follow after — For incompressible fluid get only the "afterflow" term"

May 8] We had, for small disturbances:

$$\nabla^2 p - \frac{1}{c^2}\ddot{p} = 0$$

$$p \sim \frac{f(r-ct)}{r} \quad \text{for spherical case.}$$

$$\text{radial velocity } u = p/\rho_0 c_0 + \frac{1}{r^2}\int f\,dt$$

$$\text{Energy} = \rho u^2 c$$

if afterflow is negligible $\propto \frac{1}{r^2}$

1st approximation "hydraulic" approx.

stationary flow thru channel of changing width.

cross-sectional area is slowly-varying f'n of distance

flow thru a cross-section / unit time

$= \rho u f =$ constant. continuity

$$u\frac{du}{dx} = -\frac{1}{\rho}\frac{dp}{dx} \quad \text{(eq. of motion)}$$

entropy S = constant (cons. of energy)

[for a gas: $p = \alpha \rho^{\gamma}$]

From 1st eq'n: $\frac{d\rho}{\rho} + \frac{du}{u} + \frac{df}{f} = 0$ ∴ local sound velocity $= \sqrt{\frac{dp}{d\rho}}$

$dp = + d\rho\, c^2$ ← by def.

$= - \rho\, u\, du$ (c a f'n of x).

$\therefore \frac{d\rho}{\rho} = - \frac{u\, du}{c^2}$.

$\therefore \left(- \frac{u}{c^2} + \frac{1}{u} \right) du + \frac{df}{f} = 0$

$\frac{du}{u} \left(1 - \frac{u^2}{c^2} \right) + \frac{df}{f} = 0$

if $u \ll c$: $\frac{du}{u} + \frac{df}{f} = 0$ $\quad uf = \text{const.}$

(incompressible approx)

if u is increased, change in f produces bigger change in velocity than before;

if $u > c$, increase in cross section causes <u>increase</u> in velocity of gas !

From $\frac{1}{\rho} \frac{\partial p}{\partial x} = - u \frac{du}{dx}$

get $\int \frac{dp}{\rho} = - \frac{1}{2} u^2 = w$

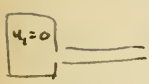
For gases $\int \frac{\alpha\gamma \rho^{\gamma-1} \cdot d\rho}{\rho} = \alpha\gamma \int \rho^{\gamma-2} d\rho$

$= \alpha\gamma \frac{\rho^{\gamma-1}}{\gamma-1} = - \frac{1}{2} u^2 = w$

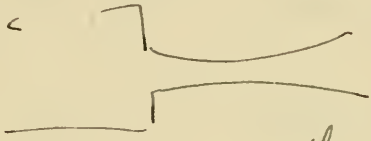
$$\therefore \quad w = \frac{\alpha \gamma \rho^{\gamma-1}}{\gamma-1} = \frac{d\not{p}/\not{\alpha}\rho}{\gamma-1} = \frac{c^2}{\gamma-1} \,.$$

$$\therefore \; c^2 - c_1^2 = -\tfrac{1}{2}(\gamma-1)(u^2 - u_1^2)$$

if $u_1 = 0$, $\quad c^2 + \frac{\gamma-1}{2} u^2 = c_1^2$ which is a f'n of temperature only



if $u \ll c$



velocity increases, then decreases;

For more flow: velocity reaches velocity of sound at smallest cross section, then increases (pressure is very low on right)

if $u = c$: $\quad c_2^2\left(1 + \frac{\gamma-1}{2}\right) = c_1^2$

$$c_2^2 = \frac{2}{\gamma+1} c_1^2$$

but $c^2 \sim \rho^{\gamma-1} \sim p^{\frac{\gamma-1}{\gamma}}$

$$\therefore \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1}$$

$$\therefore \left(\frac{p_2}{p_1}\right) = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} \,.$$

Another approximation. Riemann theory.

One-dimensional: $u(x)$ p, ρ etc. f'n's of x & t only.

[More special because of
More general because not
restricted to stationary state]

Continuity: $\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = -\rho \frac{\partial u}{\partial x}$

Eq Motion: $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$.

$\frac{dp}{d\rho} = c^2$ [local sound velocity]

Let $w = \int \frac{dp}{\rho}$

$$d\rho = \frac{dp}{c^2} = \rho \frac{dw}{c^2}$$

continuity: $\therefore \frac{\rho}{c^2}\frac{\partial w}{\partial t} + u \frac{\rho}{c^2} \frac{\partial w}{\partial x} = -\rho \frac{\partial u}{\partial x}$

or $\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} = -c^2 \frac{\partial u}{\partial x}$

also $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{\partial w}{\partial x}$

eq. motion:

Let $d\sigma = \frac{dw}{c} = \frac{dp}{c\rho}$

$= \frac{c\, d\rho}{\rho}$

$\therefore \sigma = \int \frac{c}{\rho} d\rho$

($c = c(\rho)$ if entropy is fixed)

or in terms of σ:

$$\begin{cases} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -c \frac{\partial \sigma}{\partial x} \\ \frac{\partial \sigma}{\partial t} + u \frac{\partial \sigma}{\partial x} = -c \frac{\partial u}{\partial x} \end{cases}$$

$$\frac{\partial}{\partial t}(u+\sigma) + u \frac{\partial}{\partial x}(u+\sigma) + c \frac{\partial}{\partial x}(u+\sigma) = 0$$

$$\frac{\partial}{\partial t}(u+\sigma) + (u+c) \frac{\partial}{\partial x}(u+\sigma) = 0$$

Let $u + \sigma = r$

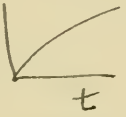
" then $r = f(x - (c+u)t)$.

i.e. it is propagated with a velocity $c+u$.

Likewise $(u-\sigma)=s$ is propagated toward $-x$ with the velocity $c-u$.

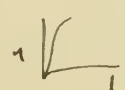
Suppose we begin moving in piston to right in a pipe with a prescribed motion. If pipe is infinite, all waves go from left to right; i.e. σ-waves

If we push to right, u at piston is $+$, and c corresponds to a somewhat compressed gas, so c increases. The later waves have a greater velocity of propagation $c+u$ than the earlier ones — will finally "catch up" with earlier ones So "steepness" of front of wave get more abrupt.

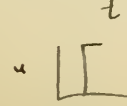


at piston.

further on



still further away.



discontinuous change of material velocity & pressure at front of wave

"shock wave"

May 11

Shock wave: surface of discontinuity of all quantities referring to gas.

discontinuity of u, p, ρ is propagated with velocity U into undisturbed gas –

Apply conservation laws to region enclosing shock front.

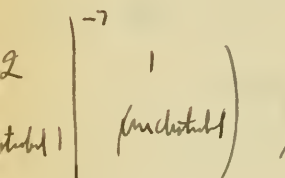
Heat conduction is large; viscosity also large (but ordinary viscosity etc. theory doesn't really apply, if wave front is very steep)

Consider shock front at rest, fluid moves ~~left~~ R to L.

i.e. velocities relative to shock front).

u_1 = velocity of gas on right $= U$

$u_2 = U - u$ (u = material velocity on left).



continuity: $\rho_1 u_1 = \rho_2 u_2$

momentum conservation: $\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2$ (pressure)

energy " : E_1 = internal energy/unit mass

$$E_1 + \tfrac{1}{2} u_1^2 - p_1/\rho_1 = E_2 + \tfrac{1}{2} u_2^2 - p_2/\rho_2$$

These are independent of material. Eq'n of state is a 4th eq'n:

$p u$ = work done by pressure/cm²/sec.

$\frac{p u}{\rho u}$ = " " per gm.

$$E = E(p, \rho).$$

Suppose we know $\rho_1 p_1 u_1 E_1$

can get explicit sol'n only for simple eq'ns of state.

In case of ideal gas:

$$E = c_v T$$

$$= \left(\frac{c_v}{R}\right)\frac{p}{\rho}$$

$\frac{c_v}{R} = 3/2$ (monatomic)

$5/2$ (diatomic)

Enthalpy: $E + p/\rho = \left(\frac{c_v}{R}+1\right)p/\rho$

$$= \frac{c_p}{R}\cdot p/\rho$$

$$= \beta\, p/\rho$$

$\frac{c_p}{R} = \frac{c_v}{R}+1$

$\beta = \gamma/\gamma-1$

From 2nd eq'n: & 1st:

$$\rho_1 u_1 (u_1 - u_2) = p_2 - p_1$$

From 3d:

$$\beta\left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) = u_1^2 - u_2^2 = (u_1-u_2)(u_1+u_2)$$

$$= \frac{p_2-p_1}{\rho_1 u_1}(u_1+u_2)$$

& get finally: $\frac{u_2}{u_1} = \frac{(2\beta-1)p_1 + p_2}{(2\beta-1)p_2 + p_1} = \frac{\rho_1}{\rho_2}$

Now $p_2 > p_1$ hence $u_2 < u_1$

(else no shock wave)

if p_2 increases indefinitely,

then $\frac{u_2}{u_1} \to \frac{1}{2\beta-1} \leftarrow \frac{\rho_1}{\rho_2}$

or $\rho_2/\rho_1 = 2\beta - 1$

Hence very high temperature

entropy $S \sim p/\rho^\gamma$ increases indefinitely

S flog p/ρ^γ.

diatomic gas, $2\beta - 1 = \frac{\gamma+1}{\gamma-1}$

$= 4, 6$

monatomic ↗ diat

$$\frac{u_1 - u_2}{u_2} = \frac{(2\beta - 2)(p_2 - p_1)}{(2\beta - 1)p_2 + p_1} = \frac{p_2 - p_1}{\rho_1 u_1^2}$$

$$\therefore (2\beta - 1)p_2 + p_1 = \rho_1 u_1^2 (2\beta - 2) + p_1$$

since $p_2 > p_1$,

must have $\rho_1 u_1^2 > \frac{\beta}{\beta - 1} p_1$

$$= \gamma p_1$$

$$u_1^2 > \frac{\gamma p_1}{\rho_1} = c_1^2 .$$

$\therefore$ shock wave velocity is always greater than sound velocity.

likewise can show that $u_2 < c_2$

velocity of shock wave relative to compressed gas is less than sound.

so sound waves ahead of shock wave are "caught" by it.

" " behind " " catch it.

if p_2 very large: $p_2 = \frac{2\beta - 2}{2\beta - 1} \rho_1 U^2$

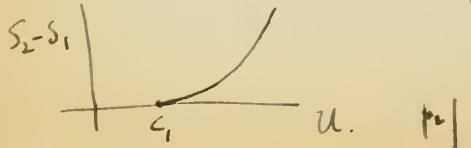
With a given starting material giving shock wave, one parameter governs the whole thing - e.g. by U.

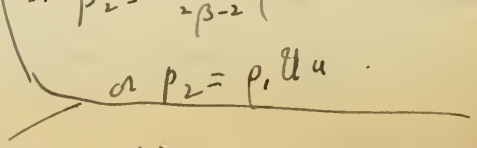
$u_2 = \frac{U}{2\beta - 1}$

$u = U - u_2 = U \cdot \frac{2\beta - 2}{2\beta - 1}$

$\therefore p_2 = \frac{2\beta - 1}{2\beta - 2} \rho_1 u^2$

or $p_2 = \rho_1 U u$.



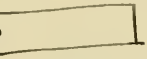


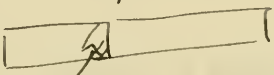
May 13 $\frac{\partial}{\partial t}(\sigma+u) + (c+u)\frac{\partial}{\partial x}(\sigma+u) = 0$

Solution is not $\sigma+u = f(x-(c+u)t)$ as was asserted, because $c+u$ is a f'n of x & t.

But a value of $\sigma+u$ at $x_0\ t_0$ is propagated in such a way that $x = x_0 + \int_{t_0}^{t} (c+u)_{t'}\, dt'$.

Detonation:

detonator → 

Chemical reaction take place rapidly, solid becoming gas + energy: propagates down length of explosive: 

High pressure, high temp.

energy E consists of heat + internal energy initially

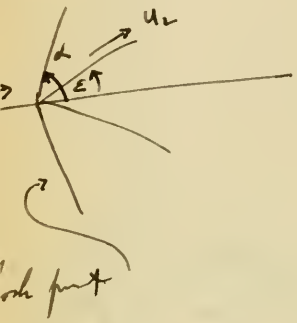
internal energy → heat energy

continuity eq'n:

May 15

wedge moving thru air with velocity greater than sound: $q_1 > c_1$

shock front is formed in front making larger angle



At shock front, velocity has two components, $\perp$ & $\parallel$ to front; we have $q_1 \tan = q_2 \tan$

$$\therefore q_1 \cos\alpha = q_2 \cos(\alpha - \varepsilon)$$

normal comp: $u_1 = q_1 \sin\alpha$

$u_2 = q_2 \sin(\alpha - \varepsilon)$

u_1 & u_2 are related by shock front eq'n's:

$$u_2 = \frac{u_1 + (2\beta - 2)\, c_1^2/q_1}{2\beta - 1} \qquad \beta = c_p/R$$

$$\text{or} \quad \frac{u_2}{u_1} = \frac{1 + (2\beta - 2)\, c_1^2/u_1^2}{2\beta - 1}$$

$$= \frac{\tan(\alpha - \varepsilon)}{\tan\alpha}$$

if ε is small: $= 1 - \dfrac{\varepsilon}{\sin\alpha \cos\alpha}$.

$$= 1 - \frac{2\beta - 2}{2\beta - 1}\left(1 - \frac{c_1^2}{u_1^2}\right)$$

$$\text{or} \quad 1 - \frac{c_1^2}{u_1^2} = \frac{2\beta - 1}{\quad} \cdot \frac{\varepsilon}{\quad},$$

1st approx: $u_1 = c_1$

and $\sin\alpha \approx \frac{c_1}{u_1}$ "Mach angle"

(shock front is locus of small waves emitted by wedge)

if $\frac{u_1}{c_1} = 1 + \delta \qquad \delta \ll 1$

$$1 - \frac{c_1^2}{u_1^2} = \frac{2\beta - 1}{2(\beta - 2)} \cdot \frac{\varepsilon}{\cos\alpha \sin\alpha} = 1 - \frac{1 - 2\delta}{\sin^2\alpha}$$

$$= 2\delta - \cos^2\alpha = \frac{\varepsilon'}{\cos\alpha} \qquad \varepsilon' = \frac{(4\beta - 1)}{(4\beta - 2)}\varepsilon$$

Can't get soln for either α too big or α too small

Minimum value of $\delta = \frac{3}{2^{5/3}} \varepsilon'^{2/3}$

---

low velocities: shock wave far from projectile.

Q.19.

$[P_{\alpha} x] = i\hbar$

$[M \times M] = i\hbar \underline{M}$

$R_0(T)$

$P(T)$

$P = P \cdot \rho \cdot \pi d l$

$I^2 R$

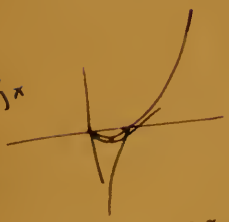
$R = R_0 \cdot \frac{l}{A} = R_0 \cdot \frac{\rho 4}{\pi d^2} =$

$\frac{I^2 4 \rho}{\pi d^2} R(T) = \rho \cdot \pi d l \cdot P(T)$

$K^2 = \frac{4 I^2}{\pi^2 d^3} = \frac{P(T)}{R_0(T)} = f(T)$

$T = t$

$x^2 \sin x$



$K = \frac{2 I}{\pi d^{3/2}}$

$x^2 \sin x = \frac{x^2}{(\sin x)^{-1}} \to \frac{2x}{-(\sin x)^{-2} \cdot \frac{1}{x}} \to -2x^2 (\sin x)^2$

$\lim_{n \to \infty} x^{1/n}$

$x^{1/n}$

$\frac{\sin x}{x^{-2}} \to \frac{\gamma x}{-2x^{-3}} \to 0$

x^{-1} $\sin x$ x x^2 . . . .

$\sin$ $e^{\frac{1}{x}}$ x e^x e^{x^2}

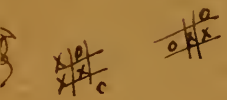
$CH_3CH_2CH(NH_2)COOH$

POWERS

4π

POWERS

$\pi \cdot r^2 = A$

ABCDEFGHIJKLMNOP

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